Friends on a Grid – Gina Pohlenz

You might have noticed that once you have a group of 13 people, at least two of them *must* have been born in the same month. Or, if you only have black and white socks lying around loosely in your drawer, you may reach into it three times and, for certain, you will end up with two socks of the same colour.

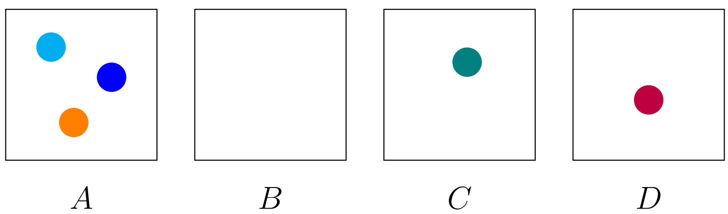
These cases are examples of Dirichlet's "pigeonhole principle", originally called the "Schubfachprinzip" (drawer principle) in German, because Dirichlet described putting pearls into drawers. If you put *more than* n pearls into n drawers in any arrangement you like, at least one drawer will contain at least two pearls!

But hold on, why is it called pigeonhole principle then? Probably this name came to be because the little drawers on old desks, that were used as a place to keep papers, look a bit like entries to dovecotes – and letters are flying in and out!

The pigeonhole principle is useful in many situations. To apply it, you need to figure out what your pearls and drawers are. Your set of disjunct "properties" are the drawers; in the sock example, the properties are "being black" and "being white". Then, having more objects (here socks) than properties means that at least two objects need to share a property.

Formally, we can state it like this:
Consider a set of objects O with $|O| = n$ and a partition of O called P, which may contain "extra" empty sets, with $|P| = m$.

> A set holds together different objects. $|O| = n$ means that there are n objects in the set O. P tells you where you put each object. Let's look at an example, here we have $n = 5$ pearls and $m = 4$ drawers, and the pearls could be sorted into the drawers in any arbitrary arrangement:



Then, if $n > m$, there is at least one set $X \in D$ with $|X| > 1$.

> P is a set of sets. So the elements of P are sets too, here they contain certain pearls. A set can also be empty! In our example above, the drawer labelled "B" is empty, while "A" contains three pearls.

We can prove the pigeonhole principle indirectly: If there was no such set X with $|X| > 1$, then each "drawer" contains either one object or no objects at all. Thus, there are maximally as many objects as there are drawers. But since we assume that there are more objects than drawers, this would be a contradiction. So at least one such set X must exist!

Now, I want to pose a problem and perhaps you can figure out how the pigeonhole principle can help. Figuring out what the drawers are is the key to the solution! I want to pose it in increasing levels of formality, pick whichever suits you the most, they all state the same problem.

> In Squaretown, all houses are built in rows and columns and every person has their own home. On a satellite image, it might look a bit like this:
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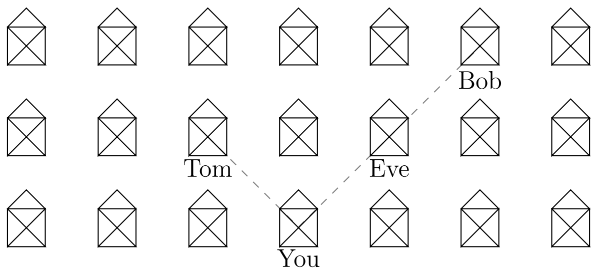
> 
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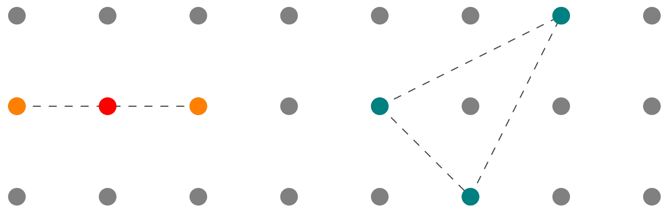
> You want to have a group of friends, but you have peculiar requirements for them: You only ever want to visit one of them at a time, your friends should want this too, and whenever two people of your group want to meet each other, they meet up in the exact middle of their houses. However, a meetup can not happen inside a house, so your meeting point should always be in an area where no house stands. In this example, Tom could be in your friend group, while Bob could not, because you'd have to meet at Eve's house, but you should meet outside! Now, how big could your friend group be with these rules?

Have a look at a sheet of grid paper. Find the maximum amount of points on that grid so that, when connecting *all* of them with each other with a line segment, the midpoints of each line segment is *not* a point of the grid paper. As an example, a line segment like the one between the two orange dots is not allowed, because the midpoint (in red) is also a grid point. The three points on the right would be a valid state though, because no midpoint is a gridpoint.



Find the maximum cardinality of a set $S \subseteq \mathbb{Z} \times \mathbb{Z}$ so that $\forall (a,b),(c,d) \in S : (a,b) \neq (c,d) \implies (\frac{a+c}{2}, \frac{b+d}{2}) \notin \mathbb{Z} \times \mathbb{Z}$.

I encourage you to draw and scribble a bit, perhaps you can get an idea for how this works or maybe even figure out how to prove it. If so, good job! If not, let's have a look at it together. I will look at the houses/line intersections/elements of $\mathbb{Z} \times \mathbb{Z}$ as if they are coordinates on the xy-plane, only allowing points that have integer coordinates.

First, you may notice that choosing the first point is completely up to you. But now, your second point is restricted. It cannot have the same parity in both coordinates as your first point! The parity of a number means whether it is even or odd. If your first point was $(0,0)$, the midpoint between it and every other point where both coordinates are even would be a grid point as well. Now we have a theory, but we need to test it.

So we propose: *If and only if each of the coordinates of one point has the same parity as the corresponding coordinate of a second point, their midpoint will be a grid point.*

We only need to check whether this argument about parity holds for one coordinate, since the midpoint will be a grid point exactly if the average of the first and second coordinate respectively is an integer. So if the idea is true for one coordinate, it will immediately hold for the other one too.

Now let's assume that for two points $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ and $(c, d) \in \mathbb{Z} \times \mathbb{Z}$, the parity of a and c are the same. If a is even, we can express it as $2k$ for some integer k. Because we assumed that the parity is the same, c can then be expressed as $2j$ for some integer j. The midpoint will be at their average, so $\frac{2k+2j}{2}$. We can see that $\frac{2k+2j}{2} = \frac{2(k+j)}{2} = k + j$, and since k and j are integers, their sum is an integer too.

Now let us assume that the coordinates do not have the same parity. That means one is even and can be written as $2k$ for some integer k, while the other is uneven and can thus be written as $2j + 1$ for some integer j. Their midpoint will then be on the coordinate $\frac{2k+2j+1}{2} = \frac{2(k+j)+1}{2} = \frac{2(k+j)}{2} + \frac{1}{2} = k + j + \frac{1}{2}$. Since $\frac{1}{2}$ is not an integer, but k and j are, this midpoint will not land on an integer coordinate.

Now we have shown that the midpoint of two points (a, b) and (c, d) will be a grid point exactly when a and c, and b and d have the same parity.

Equipped with that knowledge we can finally build our drawers! The first drawer holds all the points where both coordinates are even. The second one holds all the points where the first is even and the second is odd, the third drawer contains the points where the first is odd and the second is even, and lastly, the fourth contains the points where both coordinates are odd.

Now we can apply the pigeonhole principle. As soon as we have five(> 4) points, at least two of them would be in the same box. But if two points are in the same box, their midpoint would be a grid point as we just showed! So there can only be a maximum of four points that meet our requirements, or, stated in the language of the first phrasing of the problem: Your friend group can only consist of four people including yourself. There are many such configurations and I am sure you can find one on your own!

Adding any fifth point or house would miss our goal, as we have shown. I think this is really fascinating, because while it is relatively easy to "feel" by trial and error that the maximum amount is four, it is probably not as obvious to see why that is true.

From now on, feel free to use this tool for maths problems and try to find suitable drawers into which you can sort pearls. Perhaps think about this: When n people shake hands with each other (however often and with whomever they want), can there ever be a moment where no two of them have shaken the same amount of hands?