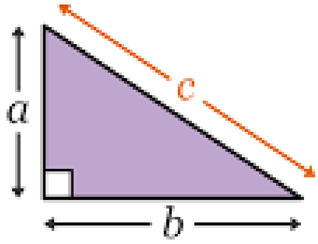
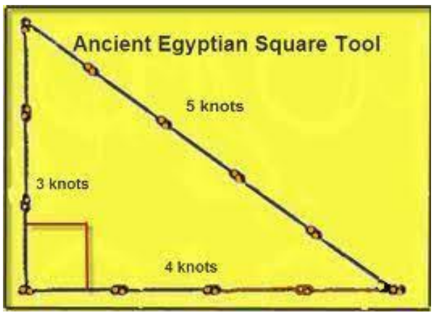
Pythagoras' Theorem- a stepping stone into Special Relativity…

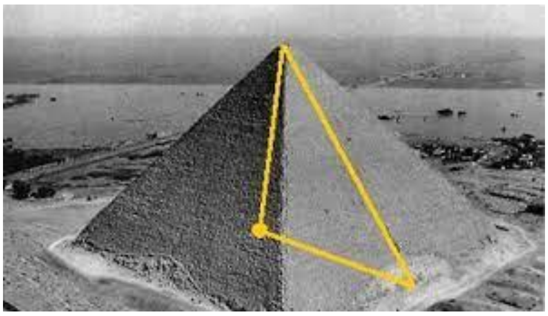
Everyone knows Pythagoras' Theorem: that for any right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the other two sides. However, not so many people know that it is more than simply an equation – from being used to build the pyramids over 4,000 years ago to now being used in pioneering space travel, its uses are broad, far-reaching and fascinating. One particularly exciting application that will be focussed on in this essay, is the fact that, by simply using Pythagoras' Theorem and Einstein's assumption that the speed of light is a constant, the factor by which time is dilated in Einstein's Theory of Special Relativity can be derived.



$$a^2 + b^2 = c^2$$

Pythagoras lived in the sixth century BC and although the theorem is credited to Pythagoras, it is actually far older. One of the earliest known applications was by the Ancient Egyptians. When building, they would mark ropes with knots 3, 4 and 5 units apart.

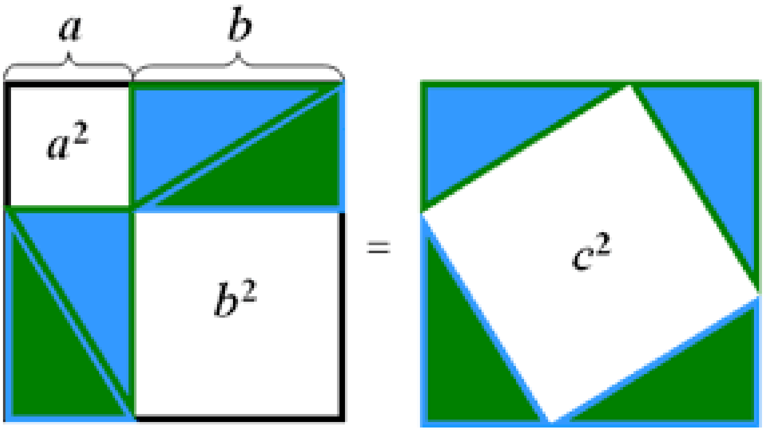




By stretching these knots around posts, the triangle formed would have a right angle opposite the hypotenuse since $3^2 + 4^2 = 5^2$. Therefore, blocks could be stacked in a way stable enough for the pyramids to be built. Interestingly, a right-angled triangle with sides in the ratio 3:4:5 is known as an Egyptian triangle and the word hypotenuse comes from the Greek for "stretched against". The 3 4 5 triangle was used as it is the simplest of all Pythagorean triples, which are special cases where all the side lengths are whole numbers.

Pythagoras' Theorem can be proved in many ways. One of the simplest is shown below:

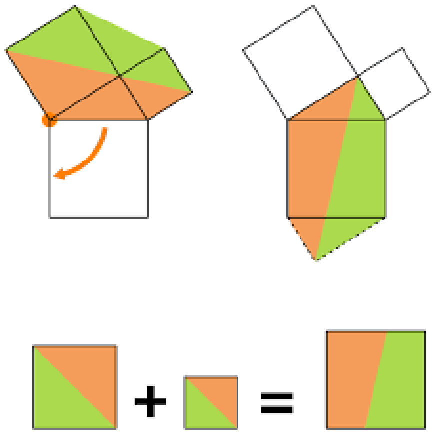
Square 1

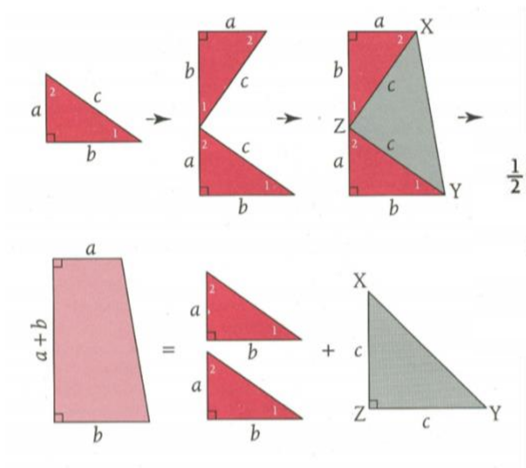


Square 2

Here, the areas of Squares 1 and 2 are the same and the right-angled triangles are all identical. For each of the right-angled triangles, let the length of the hypotenuse be c and the other two sides be a and b. The right-angled triangles in 1 can be rearranged as in 2, therefore, the white areas inside both squares are equal. The area of the white square in 2 is the square of the hypotenuse of the right-angled triangles: c^2. The areas of the two white squares in 1 are the squares of the other two sides of the right-angled triangles: a^2 and b^2. Since the white areas inside both squares are equal, $a^2 + b^2 = c^2$.

There are over 370 different proofs including the proofs shown below from Leonardo Da Vinci (left) and former US president, James Garfield (right):





$$\tfrac{1}{2}(a+b)(a+b) = 2(\tfrac{1}{2}ab) + \tfrac{1}{2}c^2$$

$$a^2 + 2ab + b^2 = 2ab + c^2$$

$$\text{Therefore, } a^2 + b^2 = c^2$$

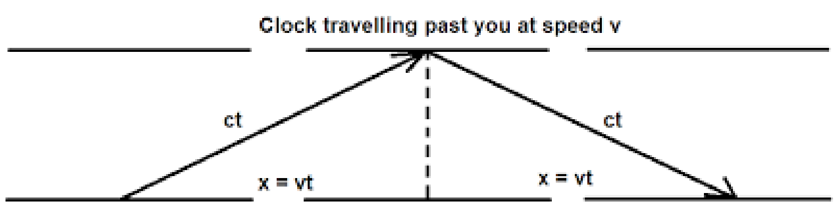
This theorem is the foundation of many mathematical ideas with, as mentioned, a particularly fascinating application being its role in the derivation of the time-dilation equation in Einstein's Theory of Special Relativity. By using Pythagoras' Theorem alongside Einstein's assumption that the speed of light is the same for everyone, Einstein discovered that time cannot be absolute. Indeed, the rate at which time passes is different for observers in different frames of reference.

Time-dilation in the Theory of Special Relativity is the slowing of time as perceived by one observer compared with another, depending on their relative motion. This can be scientifically shown using light clocks: two parallel mirrors, a distance apart, between which a light beam bounces back and forwards. One tick of the clock is equivalent to the time taken for the light beam to return to the lower mirror.

First imagine there is a light clock with two mirrors 1m apart, where one tick is the time, **t'**, that the light takes to travel from the bottom mirror to the top mirror, and back again. Therefore, the total distance is 2m.

The speed of light is a constant, **c**, and since $speed = \frac{distance}{time}$, then **t'** = $\frac{2}{c}$

Next, imagine that the light clock is on a moving train. The diagram below shows how one tick of the clock on the train looks to a stationary observer on the platform. One tick will no longer be $\frac{2}{c}$ as the light will travel further than 2m as determined from the platform because the train is moving. Let the time for one tick of the clock be 2**t**. Now, this is where Pythagoras' Theorem comes in. If time taken for one tick is 2**t,** then **t** will be the time taken for the light to travel from the bottom mirror to the top mirror. As can be seen from the diagram, to find an expression for **t**, Pythagoras' Theorem must be used. The path of the light can be represented by the hypotenuse of a right-angled triangle, where the lengths of the sides of the triangle are as follows: the vertical component of the triangle is 1m; the horizontal component of the triangle will be the displacement of the train according to the observer on the platform, which will be **vt** where **v** is the velocity of the train, and the hypotenuse will be **ct** as the speed of light is the same for everyone.



So, using Pythagoras' Theorem, $(\mathbf{ct})^2 = (1^2) + (\mathbf{vt})^2$

Rearranging this equation for **t**, gives $\mathbf{t} = \frac{1}{\sqrt{(c^2 - v^2)}}$

The time taken for one tick is 2**t** therefore, from the perspective of the person on the platform, when the clock is on a moving train, the time of one tick is $\frac{2}{\sqrt{(c^2-v^2)}}$. However, the time for 1 tick according to a person sitting on the train next to the clock is $\frac{2}{c}$.

The ratio of these two time periods for one tick shows by how much the clock on the train is running slow as measured by the observer on the platform i.e. the factor by which time dilates. The ratio of the ticks is:

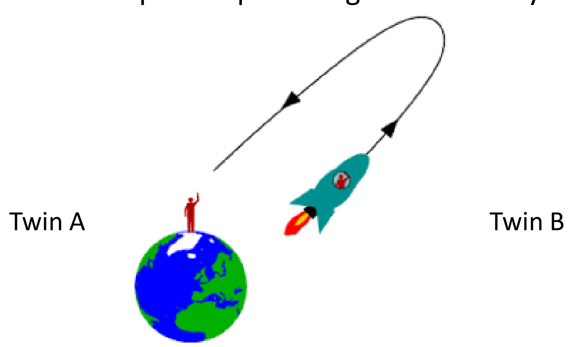
$\frac{2}{c}:\frac{2}{\sqrt{(c^2-v^2)}}$ which simplifies to 1: $\frac{1}{\sqrt{(1-\frac{v^2}{c^2})}}$ and so time has dilated by a factor $\frac{1}{\sqrt{(1-\frac{v^2}{c^2})}}$

This is a very important equation in relativity theory, and it is usually represented by the Greek letter γ.
$\gamma > 1$ when the clock is moving at less than the speed of light, because $\frac{v}{c} < 1$. When **v** is very small relative to the speed of light, γ is very close to 1 and so effect of time-dilation is negligible. However, at velocities approaching the speed of light the effect becomes significant.

Now consider the passenger on the train, sitting beside the clock. From the perspective of the person on the platform, the clock on the moving train is ticking more slowly than their clock. Therefore, time is running more slowly for the person on the train relative to the person on the platform and so the person on the platform is aging faster than the passenger on the train. As we have seen from the time dilation equation, the effect is tiny for the speeds at which trains travel. However, now imagine a person travelling away from Earth in a spaceship at a speed close to the speed of light, the effect gets magnified and from the perspective of a person on Earth, the person in space would age slower.

The slowing of time applies not just to clocks but also to everything the person in the spaceship does. From the perspective of the person on Earth it is as if the person on the spaceship is living their life in slow motion. From the perspective of the person on the spaceship, nothing has changed, but when they return to Earth, they will be younger than the person on the Earth.

However, here there is a paradox – the so-called Twin Paradox. Imagine two twins, A and B, living on Earth. Twin A stays on Earth while twin B takes a round trip in a spaceship travelling at nearly the speed of light to a distant star. When twin B returns to Earth, thanks to time dilation, they will be younger than twin A, who stayed on Earth. The paradox is that, from the perspective of twin B, they would see themselves as stationary and see twin A on Earth moving away from them. Therefore, they should appear to age more slowly. However, that is not the case because when special relativity talks about relative motion, it refers to motion at a constant speed in a straight line but, that is not what is happening here. The twins were together to start with, so twin B in the spaceship had to accelerate off and then turn around to return to Earth before decelerating to land on Earth. When the maths is worked out to account for the acceleration and deceleration, twin B in the spaceship does age more slowly than twin A who remained on Earth.



Moving away from thought-experiments, a real-world example of time dilation and proof of Einstein's Theory of Relativity is in the case of muons.

Muons are ephemeral elementary particles that can travel close to the speed of light and are created by the collision of cosmic rays with Earth's upper atmosphere, about 10km above the Earth's surface. Due to their instability, muons decay after an average lifetime of 2.2 μs. Even at the speed of light, given their short lifetime, muons would not be able to travel much further than 600m before decaying and so should not be able to last long enough to reach the surface of the Earth; however, many of them do. This is because muons can travel very close to the speed of light. At this speed time-dilation can extend their lifetime by a factor great enough that they are able to reach the Earth's surface.

Now let's look at it from the muon's perspective - they only live on average for 2.2 μs, so how do they travel the 10km or so to get to the Earth's surface? From the muon's perspective it is the Earth and its atmosphere that moves towards the muon and as with the stretching of time, Einstein showed that space is malleable too. Einstein's Theory of Relativity states that objects contract when they move. In the case of the muon travelling close to the speed of light, the distance it must travel to reach the Earth's surface contracts by enough to enable the muon to reach the Earth's surface in 2.2 μs.

In 1977, time-dilation experiments were carried out at CERN to try and prove Einstein's Theory by measuring the lifetime of muons sent around a loop in the CERN Muon Storage Ring. This revealed that the value for the extended lifetime of a moving muon was in accordance with Special Relativity.

The discovery of time-dilation has been immensely significant in developing our understanding of our universe and the physics that dictates it. The time-dilation equation is essential in space travel due to the high speeds of spacecraft and the precision required in their journey's. Also, in the case of GPS satellites used for navigation on Earth, timing errors of just a few nanoseconds can lead to positioning errors of hundreds of meters. Therefore, these GPS systems must be programmed to account for time-dilation. Hence, the time-dilation equation has critical uses in everyday life, and the time-dilation equation itself would not exist if not for Pythagoras' Theorem.

Pythagoras' Theorem is an extremely important mathematical proof and it is incredible that by simply using this Theorem and the fact that the speed of light is a constant, an equation can be derived to predict the amount by which time is different for observers moving relative to each other. The fact that we are able to calculate how time dilates, and distances contract by using the time-dilation equation may help us to travel further into space than ever before. If we could develop the technology to travel close to the speed of light, combined with the knowledge of physics and mathematics that we now have, the boundaries of our world will be opened to new horizons.

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