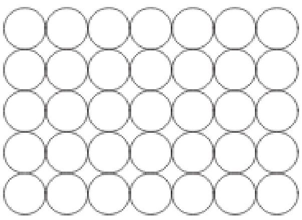
Sphere Packing

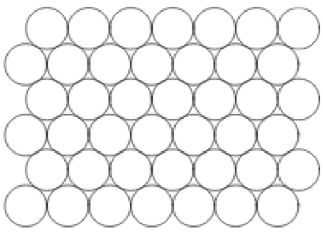
In 1900, famous mathematician David Hilbert (of infinite hotel paradox fame), set out a list of 23 unsolved problems which would go on to greatly influence the work of 20th century mathematicians. As one might expect, these questions seem largely inscrutable and verbose to those of us who have not studied mathematics at an advanced level; they were, after all, meant for the consideration of Hilbert's contemporaries.

However, the last part of question 18 reads simply: "How can one arrange most densely in space an infinite number of spheres" [1], a question that seems quite intuitive to ask. This can be explored by anyone, even a child, perhaps wanting to know how many marbles can fit into a box. Indeed, this was not the first time such a question was posed [2], as Hilbert's problem was an extension of the Kepler conjecture, first dating back to 1611.

In solving such problems it is often beneficial to reduce them into a simpler version, the solution of which may provide insight into the original. Instead of jumping straight into packing spheres, we can first consider the equivalent problem of circles.



square packing



hexagonal packing

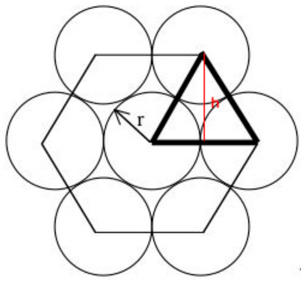
The obvious answer here is that the densest packing is hexagonal, when the circles are arranged so that each touches 6 others. This density can be found quite easily by considering the equilateral triangle connecting the

[1] The entirety of question 18 concerns "Building up of space from congruent polyhedra". This, in Hilbert's words asks: "Is there in n-dimensional euclidean space also only a finite number of essentially different kinds of groups of motions with a fundamental region?" among other questions. This refers to whether certain facts that are true in any dimension of hyperbolic and elliptic space are the same for Euclidean space. This is beyond the scope of this essay, however all parts of question 18 have now been proved.

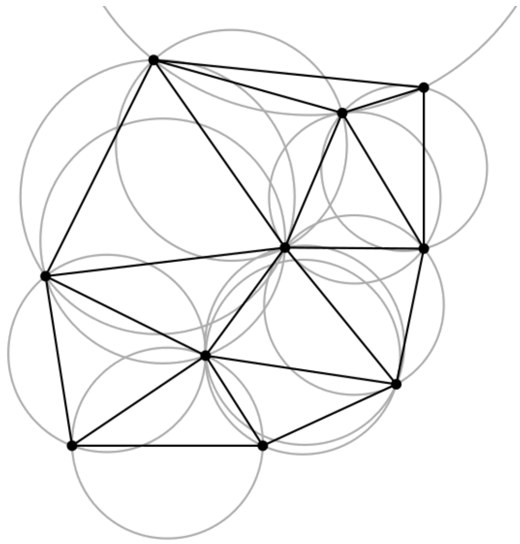
[2] It is often said that this was asked by Sir Walter Raleigh to his navigational instructor, a mathematician named Thomas Harriot. While it is true that Raleigh asked about the stacking of cannonballs, he mentioned nothing about the density of this packing. He asked Harriot:

- Given a number of cannonballs to be stacked above a triangular, square or rectangular base, how large should the base be?
- Given a pyramid of cannonballs, calculate the number of balls in the pile.

These are interesting questions (an explanation can be found at https://www.mathematicalwhetstones.com/uploads/5/4/9/9/54991295/blog_2_sir_walter_raleighs_cannon_balls.pdf) however it was Harriot himself who was not content to stop there and pursued the matter further in his correspondence with Kepler.



centres of three neighbouring circles. If each circle has a radius of length 1, the triangle has sides of length 2. Using the formula for the area of an equilateral triangle, the area is √3. There are three sectors with a radius of 1 and an angle of $\pi/3$ radians. Altogether this gives an area of $\pi/2$, and dividing by our total area of √3 gives a density of around 0.91.



Proving that this is the most optimal packing is tricky but not impossible.[3] This uses something called Delaunay triangulation. We first have to define a "circle configuration", C, the set of points of the centres of unit circles. The distance between these points has to be greater than or equal to 2. The Delaunay triangulation of C, DT(C), is a triangulation of the points such that the circumcircle (the circle which touches all of the vertices of the triangle) does not have any other points from C inside it.

This has some limitations: there is, somewhat obviously, no Delaunay triangulation for points in a straight line and it is not unique for a set of 4 points, there are always two ways to do it. However we can eliminate these by only considering what are defined as "saturated" configurations. A configuration is saturated if it is not a proper subset of another configuration (meaning its elements are not entirely contained within another configuration). We only need to consider saturated configurations as these have higher densities compared to non saturated configurations. They will always have a unique Delaunay triangulation so we can proceed with the proof.

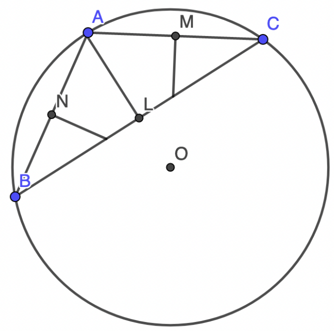
First we must show that the largest internal angle for a triangle ABC in DT(C) is between 60 degrees and 120 degrees, more precisely:

$$\pi/3 \leqslant \theta < 2\pi/3 \text{ (in radians)}$$

The largest internal angle is always bigger than or equal to $\pi/3$ radians so we only have to prove $\theta < 2\pi/3$.

[3] This was first rigorously proved by Toth in 1940, however the paper he wrote was 47 pages long. The proof shown in this essay is the 2010 proof by Hai-Chau Chang*and Lih-Chung Wang

If we assume the opposite, that the largest internal angle is greater than or equal to $2\pi/3$, this leads to a contradiction.



If the largest internal angle is greater than or equal to $2\pi/3$, the circumcentre of the triangle can be added to the circle configuration C, meaning it is not saturated.
This is proved using the inequality:

$$2R = \frac{\overline{BC}}{\sin A} \geq \frac{2}{\sin A} \geq 4.$$

Where R is the circumradius and A is the smallest internal angle. This uses the sine rule and the facts that sinA has to be ½ or less (the angle A must be at most $\pi/6$ if the largest angle is greater than or equal to $2\pi/3$ since the internal angles must sum to π) and that the distance between two points has to be at least 2 or more (as we initially assumed we were using unit circles). The contradiction means that $\theta < 2\pi/3$.

The area of a sector in radians is ½ $r^2\theta$, and as we used r=1, it is ½θ for the area of the circle inside the triangle. We can calculate the density of the triangle ABC by:

(½ A+½ B+½ C)/Area of ABC = ½ (A+B+C)/Area of ABC = $(\pi/2)$/Area of ABC

Next we must show that this density is always less than or equal to $\pi/\sqrt{12}$ (our calculated density of 0.91). If we say B is the largest internal angle, we know from our above proof that B has to be greater than or equal to $\pi/3$ and strictly less than $2\pi/3$. This means the minimum value of sinB is √3 (sin$\pi/3$). We know the minimum lengths AB and BC must be 2 (again using our definition of the unit circle). This means we can set an inequality for the area of the triangle:

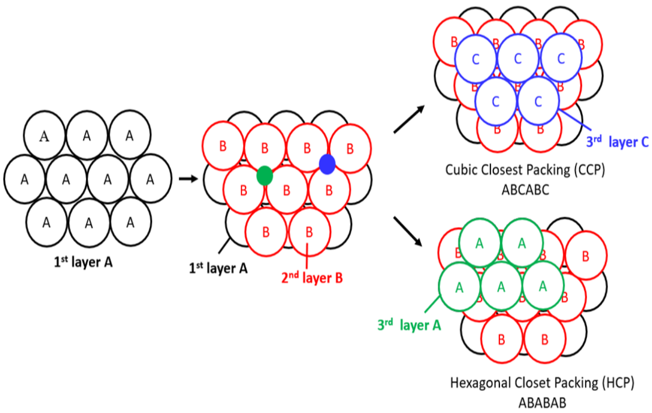
$$\text{the area of } \Delta ABC = \frac{1}{2}\overline{AB} \cdot \overline{BC} \cdot \sin B \geq \frac{1}{2} \cdot 2 \cdot 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}.$$

Therefore we can put √3 as the denominator of the equation we found previously as the minimum area of the triangle. When the density of the triangle is the highest, this area is the lowest it can be. So we can say that:

$$\text{the density of } \Delta ABC = \frac{\pi/2}{\text{the area of } \Delta ABC} \leq \frac{\pi}{\sqrt{12}}.$$

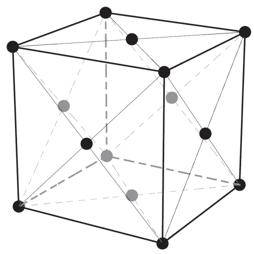
Or that the maximum density is 0.91.

But, returning back to the 3D version of this problem: what is the densest way to pack spheres? As it happens, Kepler did have an answer for this, believing that the most optimal packing was 74%, that is, if a container was packed with spheres in this way, 74% of its volume would be occupied by them. The catch is that while this seemed like the obvious answer, it resisted all attempts at a genuine proof.

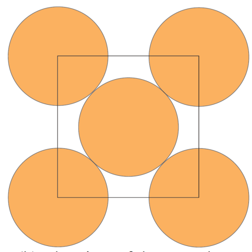


What is this packing? It is called "close packing"[4]. You can visualise this by imagining a layer of compact spheres, with no space to move around between them. In the middle of three spheres which are touching will be a gap, and the next layer of spheres will be built by aligning the centre of the new spheres above this gap. This is famously used by greengrocers to stack fruit!

The density of such an arrangement can be calculated by imagining a tessellation (the covering a surface using geometric shapes, leaving no spaces) of the 3D space with cubes. If this was overlain on the close packing of spheres as described above, a typical "unit" cube could be defined (shown in the diagram in (a)) as one where the vertices of the cube are the centre of a sphere each.



(a)



(b) The view of the cut spheres by the plane of one face

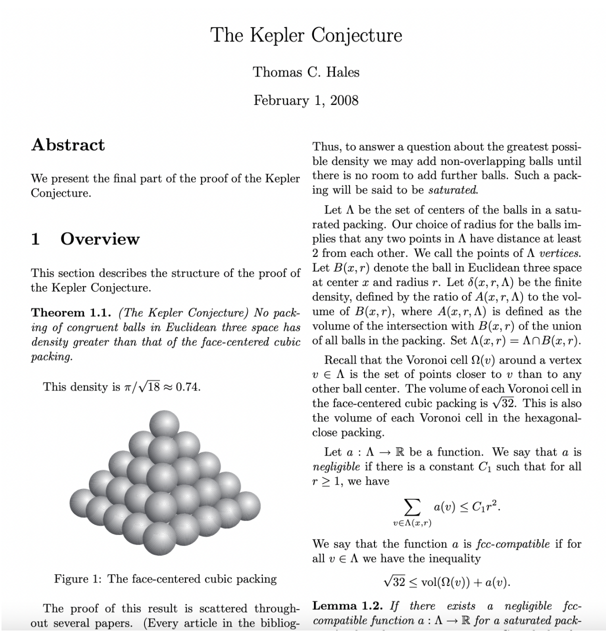
This leaves a ⅛ of a sphere inside the cube at each vertex, and as there are 8, it sums to a full sphere in the cube. Next, there will also be a sphere with a centre in the middle of each face leaving ½ of a sphere inside the cube. There are 6 faces, so this leaves 3 full spheres inside. If the radius of each sphere is taken to be 1, the length of the diagonal of the unit cube is 4 (this can be easily deduced from diagram (b)). Using Pythagoras' theorem, the length of a side of the unit cube is $2\sqrt{2}$. Therefore we have all the information needed to calculate the density. The volume of the four spheres is $4(^4/_3\pi)$ and the volume of the cube is $(2\sqrt{2})^3$. Dividing the volume of the spheres by that of the cube gives the density $^{\pi}/_{3\sqrt{2}}$, approximately 0.74.

[4] This means that the packing is not unique; there is a family of solutions to the problem. The diagram shows two different packings, but their density is equivalent. In all such packings each sphere touches 12 others.

That this was the most dense arrangement was proved in 1998, almost 400 years after Kepler first conjectured it. Many great minds attempted a proof before this of course, including the ubiquitous figure in maths, Gauss, who in 1831, did in fact manage to prove that this was true if the spheres are arranged in a regular lattice. However, the issue of whether there was some irregular arrangement which was denser still remained. Certainly, some irregular arrangements were denser over a small volume, but extending this to a larger volume seemed to always reduce the density.

However, a breakthrough was made in 1953 by the mathematician László Fejes Tóth who discovered that there were a finite number of such irregular arrangements. This meant that it was possible to simply check the density of all of these arrangements, comparing them to our figure of 74%. If all of them were less than this then it could be proved! Doing this by hand, however, would be quite an undertaking and Tóth suggested a computer could be used.

This is indeed what Thomas Hales did in 1998, using a process called linear programming to check 5,000 different configurations of spheres. However, saying that Hales simply typed in a few commands, relaxing while a computer laboriously churned out a proof seems disingenuous. He published several papers explaining the process and outlining the non-computer parts of the proof, the longest of which (around 100 pages) is linked in the sources of this essay. The whole process took 6 years and even then, a panel of referees of the proof claimed they were only 99% sure it was correct. Officially, the proof was only accepted in 2017, after Hales wrote an extremely formal proof: one which could be checked using software which automatically checks proofs.



The Kepler Conjecture

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February 1, 2008

Abstract

We present the final part of the proof of the Kepler Conjecture.

1 Overview

This section describes the structure of the proof of the Kepler Conjecture.

Theorem 1.1. *(The Kepler Conjecture) No packing of congruent balls in Euclidean three space has density greater than that of the face-centered cubic packing.*

This density is $\pi/\sqrt{18} \approx 0.74$.

Figure 1: The face-centered cubic packing

The proof of this result is scattered throughout several papers. (Every article in the bibliog-

Thus, to answer a question about the greatest possible density we may add non-overlapping balls until there is no room to add further balls. Such a packing will be said to be *saturated.*

Let Λ be the set of centers of the balls in a saturated packing. Our choice of radius for the balls implies that any two points in Λ have distance at least 2 from each other. We call the points of Λ *vertices.* Let $B(x,r)$ denote the ball in Euclidean three space at center x and radius r. Let $\delta(x,r,\Lambda)$ be the finite density, defined by the ratio of $A(x,r,\Lambda)$ to the volume of $B(x,r)$, where $A(x,r,\Lambda)$ is defined as the volume of the intersection with $B(x,r)$ of the union of all balls in the packing. Set $\Lambda(x,r) = \Lambda \cap B(x,r)$.

Recall that the Voronoi cell $\Omega(v)$ around a vertex $v \in \Lambda$ is the set of points closer to v than to any other ball center. The volume of each Voronoi cell in the face-centered cubic packing is $\sqrt{32}$. This is also the volume of each Voronoi cell in the hexagonal-close packing.

Let $a : \Lambda \to \mathbb{R}$ be a function. We say that a is *negligible* if there is a constant C_1 such that for all $r \geq 1$, we have

$$\sum_{v \in \Lambda(x,r)} a(v) \leq C_1 r^2.$$

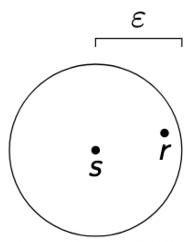
We say that the function a is *fcc-compatible* if for all $v \in \Lambda$ we have the inequality

$$\sqrt{32} \leq \mathrm{vol}(\Omega(v)) + a(v).$$

Lemma 1.2. *If there exists a negligible fcc-compatible function $a : \Lambda \to \mathbb{R}$ for a saturated pack-*

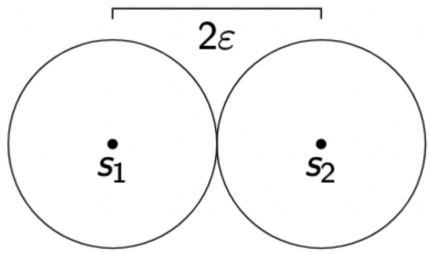
That aside, there is more to sphere packing than the packing of identical, 2D and 3D spheres only. As mathematicians are wont to do, this problem has been generalised into asking about the most optimal way to pack in any dimension, the packing of different sized spheres and more. While such generalisations are important in themselves as they add to our knowledge of mathematics, they do also have applications for the real world.

For example, the questions about higher dimensions can be used in information theory. This is because anything you can measure using n numbers can be classed as a point in n dimensional space. So if something is defined using a set of 8 measurements, it can be defined as a point in 8D.



It then becomes useful to define distance in higher dimensions, which is the same as Pythagoras' theorem, but with extra coordinates added. If you want to send signals, such as by radio, you must make sure that they are a certain distance apart, as they could create "noise" when the signals are confused, which is the main problem in communication.

In the diagram, ε is the noise level, s is the signal sent, and r is the signal received. Each signal in s should be kept a distance of 2ε apart to stop the received message getting confused. This is the same as packing spheres in whatever dimension the signal is sent!



It seems incredible to think that Kepler's conjecture
from 1611, a question one could reasonably explain to a child, could be so difficult to prove or lead to such extraordinary discoveries. Yet this is the nature of mathematics, to find the complexity in the seemingly obvious and to find creative solutions for such tricky problems.

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