The differing impacts of differential equations on the economy

The economy is complicated and in order to provide the best outcomes for all parties involved, occasionally mathematical models have to be used. Many of which include differential equations which can be quite useful in modelling real life situations, however the use of differential equations especially in these mathematical models does more harm than good. In this essay I will be describing two ways which differential equations are used in economic models, one having a positive effect on the macroeconomy and one that possibly may have caused one of the biggest financial crisis in human history.

Modelling economic growth using differential equation

The Solow-Swan growth model is a macroeconomic tool which can be used to explain and predict long run economic growth. Initially, developed by Robert Solow and Trevor Swan in the 1950's and 1960's it has proven to be one of the most influential models in the field of macroeconomics. The model itself is based on several key assumptions about the economy which in turn allow economists to create equations and analyse the relationships between various economic variables such as capital accumulation, labour or population growth and increases in productivity.

Mathematically, the Solow-Swan model is a nonlinear system which consists of a single ordinary differential equation that models the amount of capital per person throughout a time period. The key components are a production function (Cobb-Douglas production function) and the two factors of production being capital and labour growth. The model works on the assumption that to create a single good or service there needs to be two inputs being labour and capital. Capital is the stock of physical goods that are used to produce certain goods and services. In addition to this another assumption used is the idea that each additional unit of capital contributes less and less to production than the previous unit which is called diminishing returns to capital.

One of the key equations in the Solow-Swan model is the Cobb-Douglas production function, which shows how much output can be produced with a given set of inputs. The production function is typically expressed as:

$$Q = Y(t) = F(K(t), L(t))$$

Let Q and $Y(t)$ be the annual quantity of goods and services produced using K units of capital and L units of labour at time t. Two assumptions are made such that the production function Q is twice differentiable in capital and labour and the function is linearly homogeneous which in economic terms means the production function has constant returns to scale, meaning that doubling the inputs will double the output giving the following equations:

$$k = \frac{K}{L} \text{ and } q = \frac{Q}{L} = f(k)$$

Where k is the capital per worker and q is the output per worker. This gives us the rate of change of capital stock which sets up Solow's differential equation:

$$Rate\ of\ change\ of\ capital\ stock = Rate\ of\ investment - Rate\ of\ depreciation$$

$\frac{dk}{dt} = sf(k) - \sigma k$

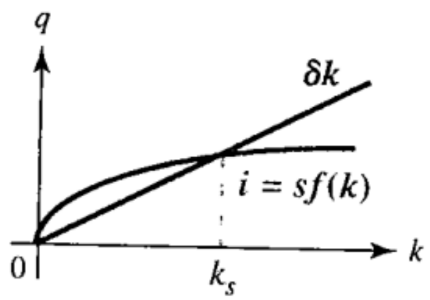
The rate of investment includes s which is a constant defining the savings rate which tends to vary depending on the economic climate. In a recession people save more so s would be higher and in times of economic prosperity s would be lower. The rate of depreciation includes σ which is the depreciation rate defined by how much capital stock depreciates annually since they may wear out or become obsolete due to other newer technologies.

The equation shows that the change in the capital stock over time is equal to the net investment in the economy $sf(k)$ minus the amount of capital that is lost due to depreciation (δK). The model assumes that the economy is in steady state when the level of capital is constant, meaning that $\frac{dk}{dt} = 0$. In steady state, the net investment in the economy is equal to the depreciation of the capital stock:

$\frac{dk}{dt} = sf(k) - \sigma k = 0$
$sf(k) = \sigma k$

The steady-state level of capital (K*) is determined by solving this equation. The steady-state level of output (Y*) is then determined by substituting the steady-state level of capital into the production function:

$Y^* = F(K^*, AL)$



The Solow-Swan model also includes a differential equation that describes the evolution of the level of labour in the economy as the population and therefore the labour force will grow at an annual rate of n:

$\frac{dL}{dt} = nL$

where $\frac{dL}{dt}$ is the change in the level of labour over time, n is the rate of population growth, and L is the level of labour. This increase in the labour force means the capital per worker will also need to increase therefore causing an annual increase in investment of $(n + \sigma)k$ in order to keep k at a constant rate.

By manipulating and solving this set of equations, economists can analyse the long-run growth rate of the economy and the effects of various policies on economic growth. For example, policies aimed at increasing the savings rate or investment in the economy can lead to higher levels of capital and output in the long run. Similarly, policies that lead to a slower rate of population growth can lead to higher levels of output per person. The Solow-Swan model is a key tool in macroeconomics that helps to explain long-run economic growth and guide economic policy.

<u>The equation that broke the economy ?</u>

The Black-Scholes model is a mathematical model that makes use of partial differential equations and geometric Brownian motion to determine the prices of different financial options such as financial derivatives. The model is based on the assumption that the price of an underlying asset follows a stochastic process and that the value of an option can be determined using key facts about it including the price of the underlying asset, the time until it expires, the strike price of the option, and the price volatility of the underlying asset.

One of the key mathematical concepts behind the Black-Scholes model is geometric Brownian motion. Geometric Brownian motion is a stochastic process in which the logarithm of a randomly varying quantity follows a Brownian motion with a drift and is commonly used to model the behaviour of complex financial assets and derivatives. In this process specifically, the rate of return of the asset is constant and normally distributed, and therefore the asset price follows a ‘random walk’ .

The formula for geometric Brownian motion is given by:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where S_t is the price of the asset at time t, μ is the expected rate of return, σ is the volatility of the asset, and sW_t is a Wiener process, which is a continuous-time stochastic process with independent and normally distributed increments.

In the Black-Scholes model, the underlying asset is assumed to follow geometric Brownian motion. This means that the price of the asset changes continuously over time and is affected by both the expected rate of return and the volatility of the asset.

The Black-Scholes equation itself is a partial differential equation and is most commonly used to determine the price of a European call option. The equation factors in key components about the call option such as the current price of the underlying asset, the strike price of the option, the time until it expires, and the volatility of the underlying asset.

The Black-Scholes equation is given by:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

Equation 1. The Black-Scholes partial differential equation describing the price of a European call or put option over time

Where V is the price of the call option, S is the price of the underlying asset, t is time, r is the risk-free interest rate, and σ is the volatility of the log returns of the underlying asset.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S}$$

Equation 2. Rewritten form of the Black-Scholes equation

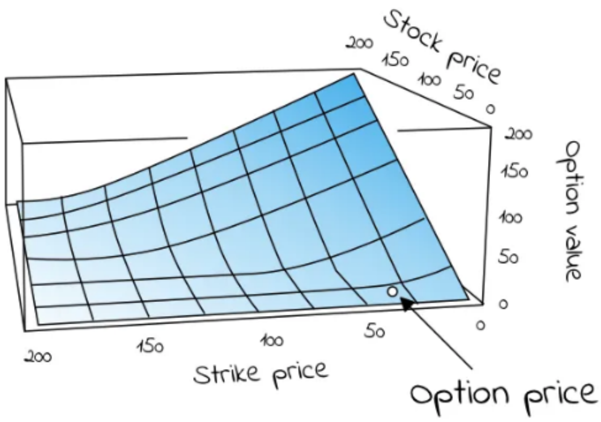


Figure 1. Visual representation of European call option price/value with respect to strike price and stock price, as calculated using the Black-Scholes equation

In the rewritten form of the equation the left side represents the change in the value of option V due to time t increasing and the convexity of the option's value relative to the price of the stock. The right hand side represents the risk free return from a long position in the option and a short position.
The first term in the Black-Scholes equation represents the time decay of the option. The second term represents the random fluctuation of the underlying asset. The third term represents the change in the option price due to changes in the underlying asset price. The fourth term represents the interest rate effect on the option price.

In order to actually receive the price of the option the Black-Scholes equation needs to be solved which can be done using various numerical techniques, such as the finite difference method, the Monte Carlo method, or the binomial tree method. The solution to the equation gives the theoretical price of the option at any given time, which can then be used to determine whether the option is overpriced or underpriced in the current market which foreshadows its role in the 2008 financial crash.

The Black-Scholes model isn't perfect though, despite it being a mathematical model, it is commonly argued that the model does not take into account the possibility of extreme events, such as market crashes or economic crisis triggered by shocks. It can also be argued that it assumes that the underlying asset follows a continuous-time stochastic process, which hardly replicates real life financial markets. These concerns were echoed in the 2008 financial crisis where mortgage backed securities were heavily mispriced due to the use of the Black-Scholes model. Despite these criticisms, the Black-Scholes model remains a widely used tool for pricing financial options and managing risk in financial markets.

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