The Math In and Around Us

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When we think of math what comes to mind for most of us is a textbook, assignments, fixed slots of time in our timetables, a subject we struggle with or anxiety due to an upcoming assessment. The least of us, however, think of math as being relaxing, peaceful and even beautiful perhaps? This essay is an attempt to bring awareness to the math that lies beneath the beauty we perceive everywhere around us from a little flower to towering trees; from a peaceful river delta to raging storms; from the intricate details in our bodies to the vastness of the universe.

Take a look at these images:

The pattern similarity is unmistakeable, isn't it?

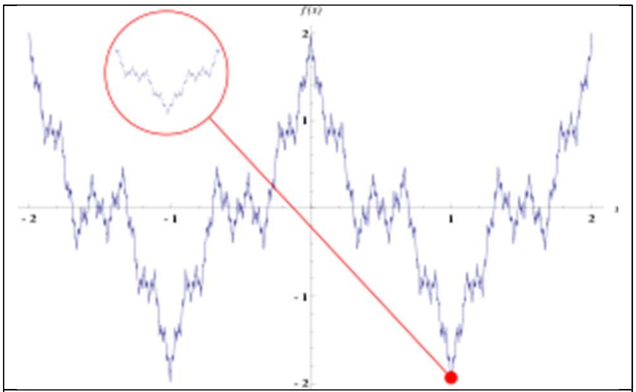
If we take a closer look at the fern leaf, it is intricately detailed with each leaf being a reduced copy of the whole. This pattern is seen everywhere in nature from branches of trees to lightning, capillary network in our bodies, dendrites in our brains and even a Romanesco broccoli in the grocery section of a supermarket.

If we can precisely determine the shape and size of objects that we have encountered in geometry, such as a square, a circle or a cube what exact shape does a fern leaf have? And we see that Euclidean geometry is unable to answer this question! Squares, rectangles, triangles and the rest may be great for modelling the shapes of man-made technology but there are few regular shapes to be found in the natural world. This begs the question; how can we describe something like a fern as a precise mathematical shape with all the rough lines and surfaces?

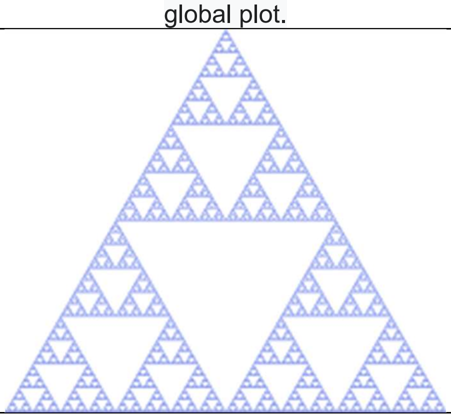
This brings us to the colourful, intriguing world of fractals.

Fractals are a never-ending pattern that repeats itself at different scales with two key characteristics: self-similarity and non-integer dimensions which I will explain in detail later.

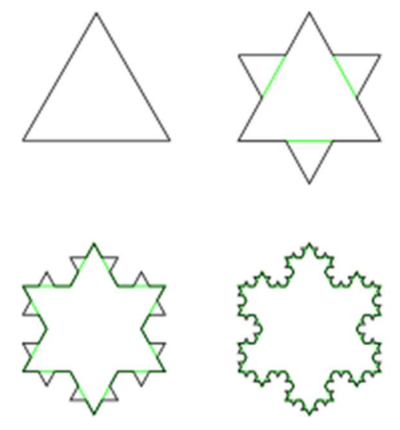
Although fractal methods were developed by many mathematicians in the 19th and 20th centuries – Georg Cantor 1845-1918; Helge von Koch 1870-1924; Waclaw Sierpinski 1882-1969; Gaston Julia 1893-1978, Weierstrass was the first to create a fractal curve. The zigzag Weierstrass constructed was so jagged, it was an unending "staccato" of corners. His line had irregular details on every possible scale – a key feature of fractal shapes. Weierstrass' line was labelled pathological and lacking all common sense as it defied the mathematically well-established method of calculus that had achieved infallible reputation over the previous centuries. The key to solving this problem was only found with modern computing methods.



Plot of Weierstrass function over the interval [−2, 2]. Like some other fractals, the function exhibits self-similarity: every zoom (red circle) is similar to the global plot.



Sierpinksi Triangle
a fractal attractive fixed set with the overall shape of an equilateral triangle, subdivided recursively into smaller equilateral triangles.



The first 4 iterations of the Koch curve
a fractal curve and one of the earliest fractals to have been described.

It was, however, Benoit Mandelbrot who coined the term and popularised it with the creation of the Mandelbrot Set by performing iterations on a simple equation thousands or millions of times. It is no wonder, therefore, that he could only create the Mandelbrot Set, and was at a colossal advantage compared with his predecessors, when he gained access to the immense computing power at IBM in the 1960s.

The term fractal was coined by Mandelbrot as he famously thumbed through his son's Latin dictionary one afternoon and found the Latin adjective "fractus" from the Latin verb "frangare" which means broken or fractured and resonated with the English term fraction. In his own words:

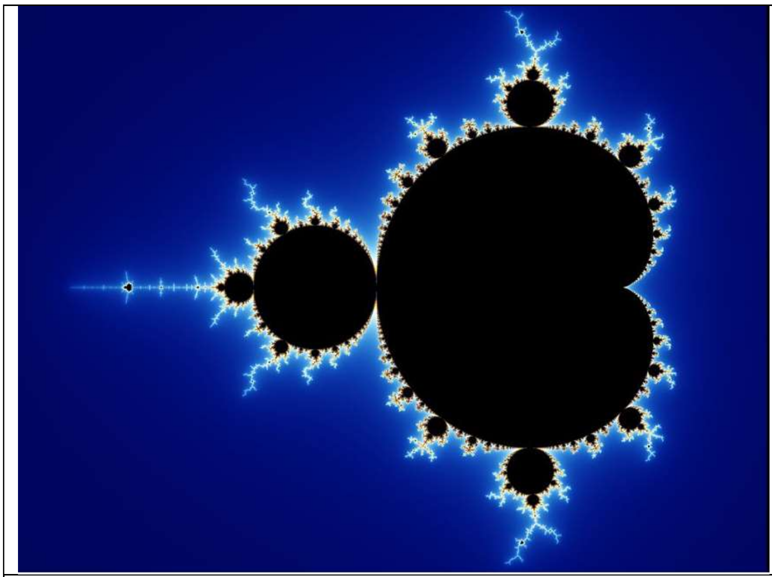
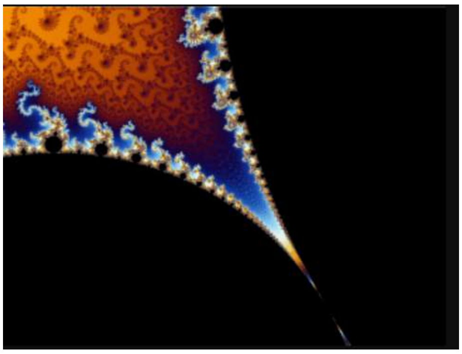
"I found myself, in other words, constructing a geometry…of things which had no geometry."

"Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line."
- Benoit Mandelbrot, 1924-2010

The Mandelbrot Set is the set of points on a complex plane. To build the Mandelbrot set an algorithm based on the recursive formula

$Z_{n+1} = Z_n^2 + C$ (where Z is a complex number and Z_0=0)

is used separating the points of the complex plane into two categories - points inside and points outside the plane, i.e., for different values of C, the trajectories either stay near the origin or "escape". The points of the Mandelbrot set have been coloured black and are points that are not in the Escape set. The colours of the points outside depend on how many iterations have been required to determine that they are outside the set.

	
Mandelbrot Set	Self-similar copy

The Mandelbrot Set is not only aesthetically appealing but was at the time a dramatic demonstration that extremely simple rules can produce extremely complex results (up until this time, you needed complex rules to produce complex results). When you zoom into it, the reduced copies are not identical but only similar (Quasi self-similarity in contrast to exact similarity in the Koch snowflake).

The common techniques to generate fractals include Iterated function systems, L-systems, Escape-time fractals and Random fractals which I will not explain here so as not to exceed the maximum number of words for this essay.

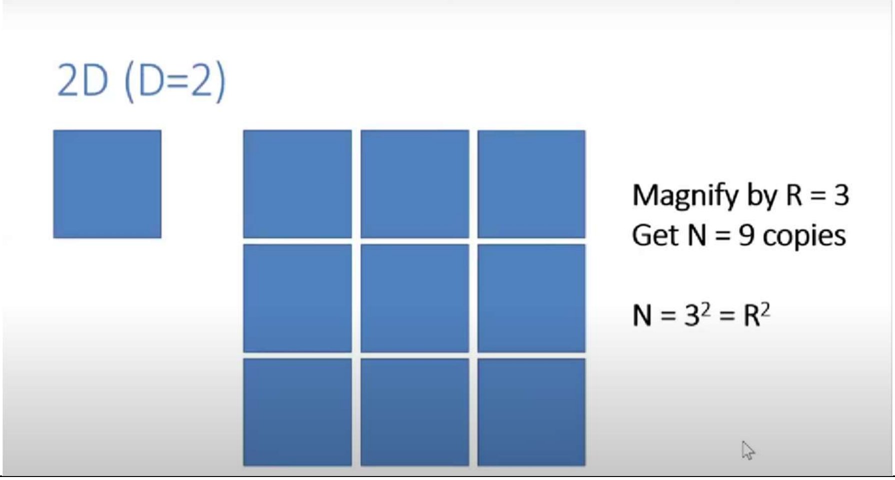
Self-similarity:

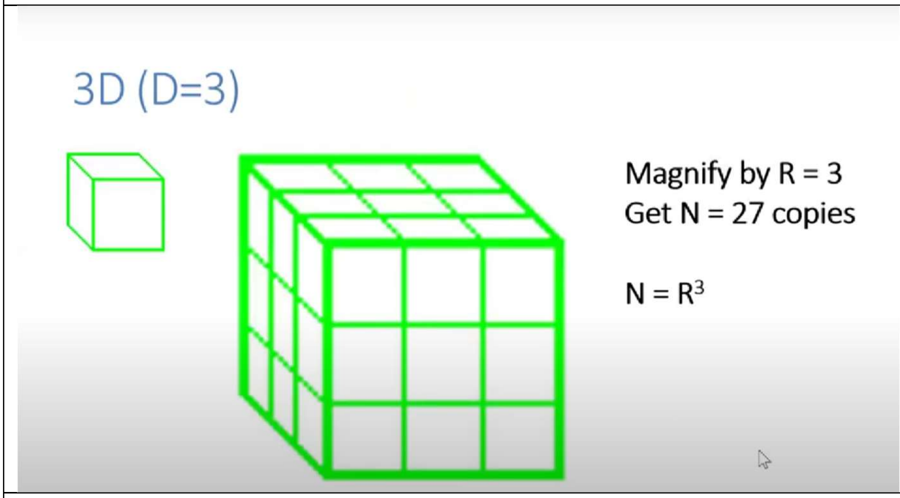
Self-similarity is symmetry across scale. It implies recursion, pattern inside of pattern. Even gigantic shapes such as the Koch curve display self-similarity as the patterns look alike even under high magnification.

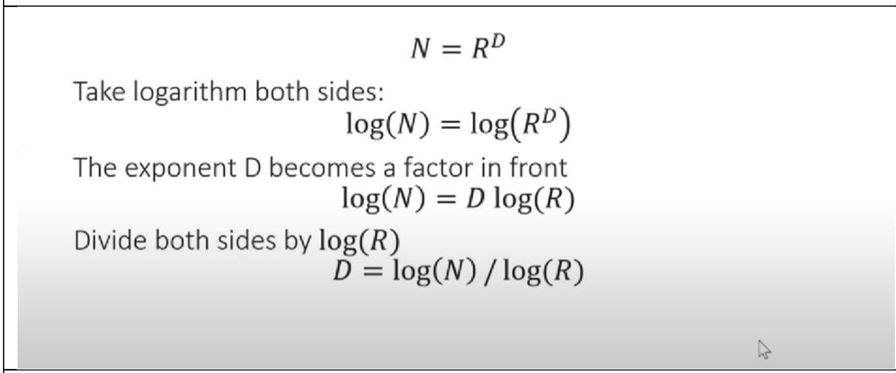
Non-Integer Fractional Dimensions:

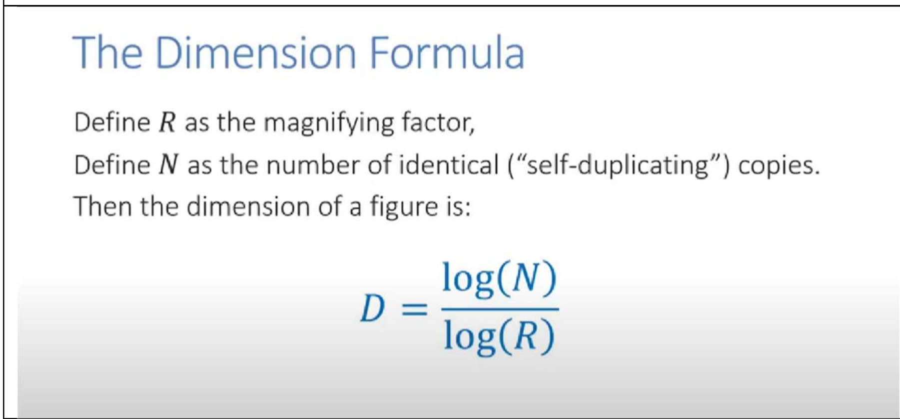
The most striking characteristic of fractals is that they are not whole dimensions of 1, 2 or 3 but somewhere in between. Intuitively, I think of dimension as a measure of how well an object fills up the space around it. This is best illustrated using a simple sheet of paper which is 2-dimensional. Compared with this sheet of paper, a sphere is 3-dimensional. Now if we crumple this sheet of paper it fills up more space than the sheet but less that the sphere and has a dimension of approximately 2.5. This principle is applied everywhere in nature. An example for this is the fact that the bronchioles in our lungs if spread out would occupy the surface area of two tennis courts but crumpling them up enables them to occupy a small volume of just two or three tennis balls! By packing such a huge surface area into our bodies, nature has ensured we get the needed oxygen supply; vital to keep us alive (lungs have a fractal dimension of 2.97). It also enables structures like the Eiffel Tower to remove weight as it gets higher, without removing structural strength.

This is how fractal dimensions are calculated:

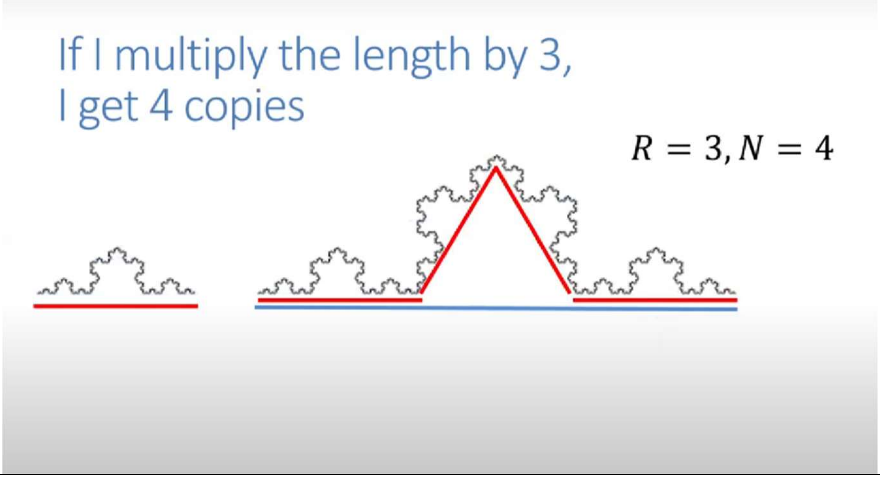


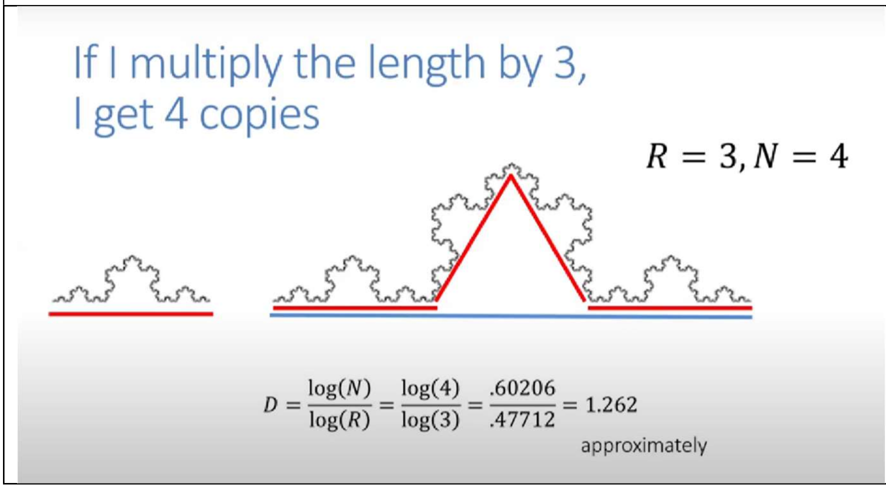






A practical example using the Koch curve:





As fractal mathematics capture the infinite complexity of nature, it helps understand many of the systems we live in better, that exhibit complex chaotic behaviour. In other words, it gives us new insight with things that are effectively impossible to predict or control like turbulence – in weather or the stock market.

Fractals in art:

When humans put fractals to intentional use in order to prove or disprove; gather empirical data or find new interconnections; postulate theories or explain concepts we call them scientists. But when humans unintentionally use fractals as a means of self-expression creating in the process a unique artifact that will keep generations to come awestruck, we call them artists.

Fractals have been used long before they were termed as such by artists in the form of Mandalas (spiritual symbols said to represent different aspects of the universe and used to enlighten the brain and heal from within) and in architecture of the ancient world. Jackson Pollock, a creator of Abstract Expressionism and one of the best-known fractal artists, has created works of art that according to neuroscientists induce stress reduction.

A fascinating study by a physicist, Richard Taylor, mesmerized by Pollock's art as a child, went on to explore the reasons why people were so drawn to his art. So, Taylor along with an environmental psychologist, Caroline Hägerhäll, ran experiments to see what the physiological response viewing these images and similar fractal geometries evoked. He measured people's skin conductance (an indirect measure of sympathetic autonomic activity that is associated with both emotion and attention) and found that they experienced a 60 percent higher stress relief when viewing computer images with a fractal dimension between 1.3 and 1.5, i.e., the fractal dimension of large, coarse patterns (coastline from a plane, main trunk of tree to Pollock's big patterns) to fine ones (dunes, rocks, branches, leaves and Pollock's fine patterns).

Further research showed an astonishing connection leading to an interesting theory by Taylor and Hägerhäll and that is, in addition to lungs, capillaries and neurons the visual system as expressed by the eye's retina branched into fractals as well. Using tracking mechanisms to see where people were focusing on projected images, the search pattern used was itself fractal! The eye first scanned the big elements in the scene and then made micro searches in smaller versions of the big scan – all in the range of 1.3 to 1.5! Even foraging patterns when tracked see this fractal pattern of search trajectories. The scientists, therefore, conclude that our visual cortex feels most at home alongside natural features we have evolved. It also shows that being surrounded by Euclidean built environments; we risk losing connection to our natural stress reducer of visual fluency. And doesn't this explain why we need to make our cities and living spaces greener and get outside more often? Or why looking at a green tree every now and then is the most natural way of relieving stress instantly?

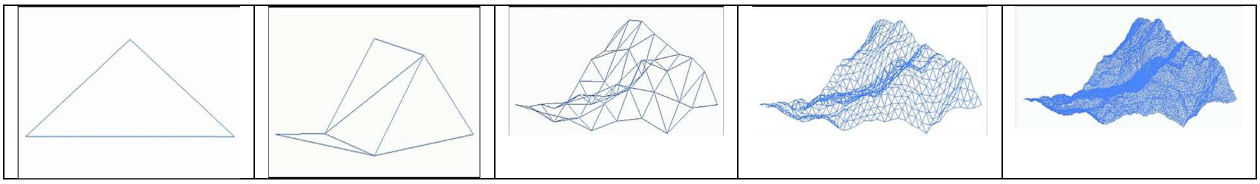
Uses of fractals:

Astrophysicists believe that the key to how stars were formed and ultimately found their place in the universe lies in the fractal nature of interstellar gas. The hierarchical structure of turbulence that shapes clouds and gives them an irregular but repetitive pattern is what could give us clues about celestial bodies.

Scientists have discovered that chromosomes depict tree like architecture and consists of mini chromosomes and can, therefore, be treated like a fractal (fractal dimension

2.34). As self-similarity can also be found in DNA sequences and can be used to solve evolutionary relationships in animals.

For the gamers among us (no pun intended), fractal forms are used to create graphical representations of textured landscapes such as mountains, coastlines etc. for computer games as they display the complexity from simplicity characteristic. The shape of the fractal can be completely achieved by a small list of mappings that defines exactly how the smaller copies are arranged to form the whole fractal. Special effects can be seen as the geography of the moon in the Return of the Jedi and to draw the outline of the famous Death Star.



Many image compression schemes use fractal algorithms to compress computer graphics files into less than a quarter of their size.

Self-similar branching patterns seem to be nature's invention to minimize friction and maximize growth (in plants or organs) or progress (in flow).

If this topic has interested you, I urge you to look up the many different fractal sets, their creation and beauty on http://fractalfoundation.org/.

This essay is only a fraction, an appetiser so to say, of the countless examples of math in us and around us as infinite forms in nature lead us to appreciate not only the obvious but further explore the unseen beauty of math that lies beneath.

References:

All images are either general Wikipedia images, my own photographs or copyrighted images used under Fair use.

Calculation of fractal dimensions taken from: https://www.youtube.com/watch?v=v273ElS1TRU

Chaos. The Amazing Science of the Unpredictable by James Gleick. I have used all relevant information from the chapter A Geometry of Nature (pages 83-118).

The Fractal Geometry of Nature by Benoit B. Mandelbrot

The Atlantic: Why are Fractals So Soothing by Florence Williams and Aeon

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