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The mathematics behind a soap film:

Minimal Surfaces

Do you remember blowing bubbles as a child, perhaps thinking that the bubbles produced were spherical because the initial soap film was a circle? But if you ever had the chance to try blowing a cube shaped bubble, you would have quickly realised it always creates a perfect sphere despite the original shape of the soap film being a square or anything else. This is because the bubble naturally tries to minimise its surface area for a given volume of air trapped within the soapy film, and due to the air pressure inside and outside the bubble pushing against each other equally, it forms a perfect sphere.

Despite the bubble forming such that it has the smallest surface area for a set volume, it is not actually a minimal surface. As you might notice, the bubble is in fact trying to collapse into a flat plane. This would form a 2-dimensional shape (called the subspace) in a 3-dimensional space (called the ambient configuration) and can easily be simulated by the soap film before you blow a bubble because the surface of the soap film is smooth and flat. Of course, in reality it will have some thickness, but it works for the purpose of visualisation as it shows that for a defined boundary, the simplest minimal surface you can have is a flat plane because it is the area of the least energy. This is a trivial type of minimal surface generally called a hypersurface where a subspace of dimensional space (n) is in an ambient configuration of a dimensional space (n+1), for example a 3-dimensional shape ($\mathbf{R}^3$) in a 4-dimensional space ($\mathbf{R}^4$). Therefore, all hypersurfaces have a codimension 1 because between $\mathbf{R}^n$ (the subspace) and $\mathbf{R}^{n+1}$ (the ambient configuration) the difference in their dimensional space is always "(n+1) – (n)" which gives "1", and is true for all hypersurfaces shown by this generalisation.

Now let's take this idea of hypersurfaces to its simplest level, a subspace $\mathbf{R}^1$ and an ambient configuration $\mathbf{R}^2$, in other words, a straight line on a graph connecting two defined points. What becomes immediately obvious? The minimal surface is the shortest distance between the enclosed space in this example, the two points. Similarly, for all set boundaries, the minimal surface has the least surface area locally and has a mean curvature of zero. This means that a minimal surface of $\mathbf{R}^3$ would satisfy Lagrange's equation which, a partial differential equation (PDE), for which all solutions are minimal surfaces. The equation for objects in a 3-dimensional space is as follows:

$$\left(1+h_v^2\right) h_{uu} - 2\, h_u\, h_v\, h_{uv} + \left(1+h_u^2\right) h_{vv} = 0$$

(Figure 1) Source: Wolfram MathWorld

This daunting equation shows that "**H** = 0" (the mean curvature of an object of $\mathbf{R}^3$ known as the Guassian curvature) for which the minimal surface is parameterised. The idea that the mean curvature (**H**) will be zero is true if the shape is a minimal surface.

We can take a simpler example and prove that a straight line of $\mathbf{R}^1$ connecting two points in an ambient configuration of $\mathbf{R}^2$ is a minimal surface by using the idea that the mean curvature is zero. This can be done by taking the general equation for a straight line and finding the second derivative, which shows the rate of change of the gradient or curvature, like so:

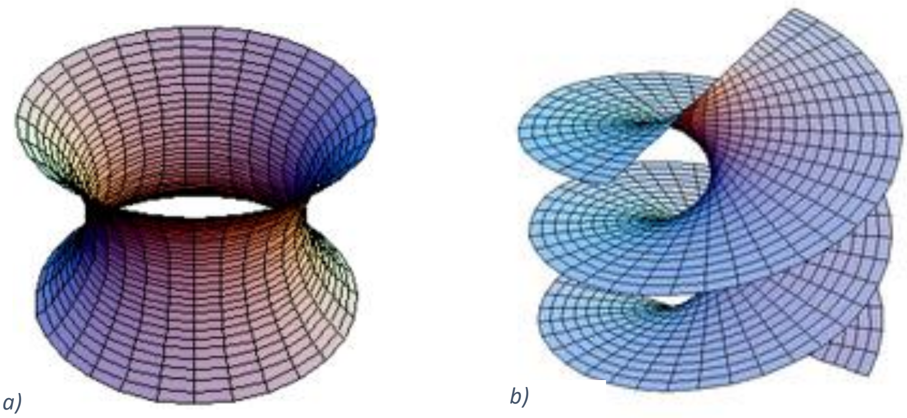
$$y = mx + c$$

$$\frac{dy}{dx} = m$$

$$\frac{d^2y}{dx^2} = 0$$

If the equation contained an exponent of "x" other than '1' or '0' (e.g. a cubic), we would clearly be left with a multiple of "m" and not '0' after finding the second derivative. Therefore, the most efficient way to connect two points on a plane of $\mathbf{R}^2$ – giving a minimal surface – is with a straight line of $\mathbf{R}^1$. This exact same idea can be applied to any other shape in higher dimensions and can therefore be used to show something is a minimal surface.

At this point, I hope you are curiously wondering what other minimal surfaces are there besides a hypersurface? Well, after much research, two more minimal surfaces were found during the 18th century, the catenoid in 1744 and the helicoid in 1776 – discovered by Euler and Meusnier respectively. These two shapes are what is known as complete minimal surfaces because they repeat and carry on endlessly. (The same is true for a plane however there are other types of minimal surfaces which aren't continuous and will be looked at later).



a) *b)*

(Figure 2) Source: Wikimedia Commons
a) Catenoid
b) Helicoid

The catenoid is defined by (where “ a > 0 ”):

$$u = \frac{1}{a}\cosh^{-1}(a\sqrt{x^2+y^2})$$

(Figure 3)

The helicoid is defined by:

$$u = \tan^{-1}(y/x)$$

(Figure 4)

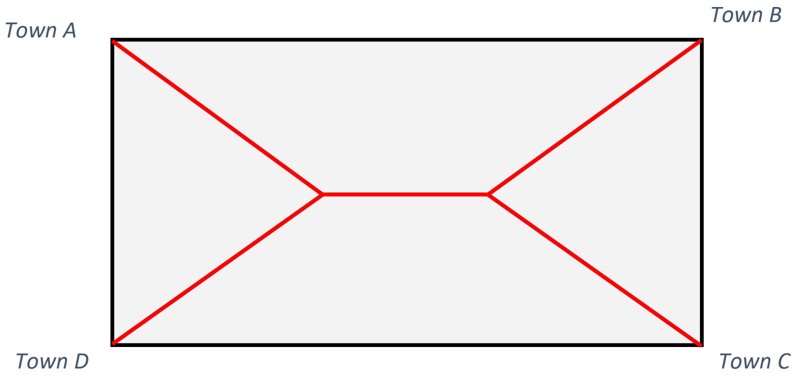
(Figures 3&4) Source: The University of Oxford

Feel free to test either of these for fun on a 3D graphing calculator to see the shapes for yourself! As you can tell, research into minimal surfaces has been ongoing for several years and is still being studied to this day with Karen Uhlenbeck winning the Abel prize for her research in geometric analysis, including minimal surfaces, in 2019. Not only do minimal surfaces look visually pleasing, they are also often used in architectural designs to create tensile roofing. This is done such that the membrane held in place by steel cables – which make up the frame – are minimal surface therefore reducing the amount of material used and creating artistic designs at the same time.

Interestingly, we find that soap film can be used to create minimal surfaces such as the helicoid and catenoid. Say we were to create a frame out of wire in the desired shape of the defined boundary and dipped it in soapy water, the shape that the bubble film makes would automatically form the minimal surface of that defined loop. This remains true given that the soap film is touching all the wire and that the wire reflects any local maximums or minimums within the defined boundary. It is amazing how nature seems to solve what are complex mathematical problems perfectly for us in this way, just by using soap film. Whilst we need the actual mathematics of the shapes produced to model and replicate these minimal surfaces, it is still enjoyable to be able to visualise them. The reason for soap film behaving in this way is down to the soap film immediately trying to create a shape within the set boundary that has the least energy relative to the boundary and therefore has a mean curvature of zero causing the soap film to reduce its surface area and form a minimal surface. For instance, we can use this idea to investigate what the minimal surface for a cube frame is and we get:

(Figure 5) Source: Wikimedia Commons

Note that this minimal surface made by the set boundary of a cube frame is not a complete minimal surface which is because this shape does not carry on infinitely, unlike the helicoid and catenoid. Similarly, it is possible to find out the minimum length of road required to connect four towns (vertices of a rectangle) on a plane $\mathbf{R}^2$, we can use minimal surfaces to solve this problem using soap film. Carrying this out, we get something like this (as shown by the red lines):



(Figure 6)

Now for the question that's been looming in our minds, is there an infinite number of minimal surfaces? Douglas and Rado were working on this problem and managed proved that there was a solution to the general case in 1931 and 1933 respectively. But their research did not discount the possibility of singularities which meant that there could still be points where their analysis becomes undefined and therefore collapses. However, in 1970, Osserman showed that a minimising solution for a restricted boundary could not have any singularities and so, there will always be a solution. This proved that all minimizing solutions will not break down at any point and thus, there is an infinite number of minimal surfaces as there is always a minimal surface for a set boundary.

The topic of minimal surfaces becomes even more complex and interesting as you dive deeper, from looking at the mathematics behind bubbles to black holes and much more. Next time you come across bubbles, I hope you take the time to try creating different minimal surfaces to really admire these magnificent shapes and see nature in action, first hand.

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