Why Newton Resorted to God: An Exploration of the Three-Body Problem and the Verlet Method

In 1687, Newton published his *Philosophiae Naturalis Principia Mathematica*, redefining the common understanding of astronomy. Newton fashioned Kepler's approximations into 'laws', providing rules to describe the motions of planets with unprecedented accuracy. Armed with Newton's newfound law of gravity, the Kepler problem (which has been amusingly referred to as the 'granddaddy of all problems in dynamical[1] systems'[2]) could now be solved analytically. However, further advances from this development, particularly concerning the motion of our solar system, proved challenging.

The Kepler Problem is a special case of the Two-Body problem, in which one considers the motion of two massive objects. For example, Kepler considered the orbit of a single planet around the sun, which he approximated to be elliptical. Newton's Law of gravitation is as below:

$$F = G\frac{m_1 m_2}{r^2}$$

Thus, the two body problem can be represented as:

$$\begin{aligned} m_1\mathbf{a}_1 &= \frac{Gm_1m_2}{r_{12}^3}\left(\mathbf{x}_2-\mathbf{x}_1\right) \\ m_2\mathbf{a}_2 &= \frac{Gm_1m_2}{r_{12}^3}\left(\mathbf{x}_1-\mathbf{x}_2\right), \end{aligned}$$

with bodies of masses m_1 and m_2, positions $\mathbf{x}_1$ and $\mathbf{x}_2$, velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, G representing the gravitational constant and $r_{12}=|\mathbf{x}_1-\mathbf{x}_2|$

With help from Newton's Law of gravitation, the Two-Body problem found a general solution. We can essentially reduce the equations to a pair of one-particle systems through integration and consideration of conservation of momentum. This leaves the problem completely able to be solved analytically. This boded well for Newton and other astronomists of the 17th century, who, in the light of common acceptance of heliocentrism, had become increasingly concerned by the prospect of the Earth crashing into the Sun or spiralling out of the solar system. However, the solar system is not a Two-Body system. Although Kepler's ellipses and consideration of only two bodies at a time provided a largely accurate approximation, it failed to consider the gravitational effect of planets upon other planets.

Let us first consider three bodies instead of two. The Three-Body problem considers the motion of three massive objects. Given Newton's Law of gravitation, it can written as follows:

$$\begin{aligned} \ddot{\mathbf{r}}_1 &= -Gm_2\frac{\mathbf{r}_1-\mathbf{r}_2}{|\mathbf{r}_1-\mathbf{r}_2|^3} - Gm_3\frac{\mathbf{r}_1-\mathbf{r}_3}{|\mathbf{r}_1-\mathbf{r}_3|^3}, \\ \ddot{\mathbf{r}}_2 &= -Gm_3\frac{\mathbf{r}_2-\mathbf{r}_3}{|\mathbf{r}_2-\mathbf{r}_3|^3} - Gm_1\frac{\mathbf{r}_2-\mathbf{r}_1}{|\mathbf{r}_2-\mathbf{r}_1|^3}, \\ \ddot{\mathbf{r}}_3 &= -Gm_1\frac{\mathbf{r}_3-\mathbf{r}_1}{|\mathbf{r}_3-\mathbf{r}_1|^3} - Gm_2\frac{\mathbf{r}_3-\mathbf{r}_2}{|\mathbf{r}_3-\mathbf{r}_2|^3}. \end{aligned}$$

*with masses m_1, m_2, and m_3, position vectors **r**, and G representing the gravitational constant.*

At first glance, these equations appear similar to those of the Two-Body problem. However, we can no longer simplify nor integrate to produce one-particle systems, which are far more easily solved. Despite the

[1] a system that changes over time and can be described using a set of differential equations

[2] Biello J, 'The Kepler Problem' <https://biello.faculty.ucdavis.edu/wp-content/uploads/sites/588/2019/01/KeplerProblem.pdf> [accessed 31 March 2023]

fact that mechanical quantities such as energy and angular momentum are still conserved, in the 1800s Henri Poincaré and Heinrich Bruns proved that there are no quantities in these equations that can be expressed as algebraic functions of the positions or velocities of the bodies (in standard Cartesian coordinates), nor mass ratios. Thus, the conclusion was drawn that the Three-Body problem cannot be solved in terms of algebraic formulas and integrals. To this day, there is no known analytical solution for the Three-Body problem.

Newton, therefore, found himself bested when confronted with the motion of three massive objects. If the Three-Body problem seemed unsolvable in a general manner, then the n-Body problem representing our solar system and other complex systems was certainly out of reach.

It was here that Newton was forced to resort to God to explain the stability of the solar system, suggesting in his *Principia* that perhaps God nudged planets back into their orbits every now and then.

Although the question of the stability of our solar system remains unsolved in a general manner, by way of numerical methods, we can calculate that our chances of flying off into deep space in the near future[3] are slim to none. One such numerical method is Verlet integration.

Verlet integration, or the Verlet Method, was first used in 1791 by Jean Baptiste Delambre and has been rediscovered multiple times since then, most recently in the 1960s by Loup Verlet for use in molecular dynamics. The Method assumes constant acceleration between timesteps. Therefore, though an approximation, given small enough timesteps, the Verlet Method remains highly accurate.

Take the following kinematic equation:

$$x = x_0 + v_0 t + \frac{1}{2}at^2 + \frac{1}{6}bt^3 + \cdots$$

*where **x** is the position, **v** is the velocity, **a** is the acceleration and **b** is the oft-forgotten jerk term*

We can then add and subtract a timestep from it[4]:

$$x(t + \Delta t) = x(t) + v(t)\Delta t + \frac{1}{2}a(t)\Delta t^2 + \frac{1}{6}b(t)\Delta t^3 + \mathcal{O}(\Delta t^4)$$

$$x(t - \Delta t) = x(t) - v(t)\Delta t + \frac{1}{2}a(t)\Delta t^2 - \frac{1}{6}b(t)\Delta t^3 + \mathcal{O}(\Delta t^4)$$

The big O notation here refers to the clumping of all terms above the fourth power to estimate error[5].

If we then sum these terms as follows:

$$x(t + \Delta t) = 2x(t) - x(t - \Delta t) + a(t)\Delta t^2 + \mathcal{O}(\Delta t^4)$$

We are left with an equation for 'stepping [the equation] forward' in time provided we know its current acceleration, current position and last position. Thus, by iteration, we can accurately predict the motion of three-or-more-body systems.

Such methods are useful when approaching deterministic chaos theory, in which a dynamical system displays seemingly random behaviour that can be attributed to slight differences in initial conditions. These systems are completely predictable, but the smallest change in starting conditions (typically measurement

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[3] The 'near future' here relates to the next 100 million years or so. However, due to the nature of chaotic systems and the impossibility of perfect accuracy of measurement, it is difficult to predict beyond this.

[4] Here, we are actually performing a Taylor Series Expansion about x(t±Δt)

[5] We leave the big O term at t^4 because when t is less than 0, the term to the fourth power will have the biggest effect of all the infinite remaining terms on the outcome.

or rounding errors) will have radical effects over time. As such, our solar system, defined by Newton's Law of Gravitation, yet unable to solved analytically, is a chaotic system.

The reasons for Newton's urgency in solving the n-Body problem were clearly demonstrated in the 2009 computational study, performed using similar mathematical methods to Verlet integration. The study attempted to predict the motion of the solar system up to 5 billion years in the future, given precise initial conditions. The only difference between each experiment was the initial position of Mercury varying by ±1 mm. Over 2000 simulations were performed. In ≈1% of these, Mercury crashed into the Sun or collided with Venus, and in one of them, Mercury disrupted the entire inner solar system. This is chaotic behaviour, where small changes have massive eventual effect.

Performing simulations on this scale, considering many massive bodies, is computationally taxing. The Fast Multipole Method, created specifically to tackle the n-Body problem, is a computational algorithm that the study likely used to increase the efficiency of approximation. Developed by Leslie Greengard and Vladimir Rokhlin in the 1980s, it is a numerical technique that can efficiently compute long-range forces in the n-Body problem. It reduces the computational complexity of the direct n-body simulation from $O(n^2)$ to $O(n \log n)$ or even $O(n)$ in some cases, making it significantly more efficient for large-scale simulations.

The Method works by approximating the effect of a cluster of particles on a distant particle as a single, multipole expansion. This approximation significantly reduces the number of pairwise interactions that need to be calculated, accelerating computation without sacrificing too much accuracy. By utilizing advanced algorithms like the Fast Multipole Method, we can simulate and analyse the behaviour of massive bodies in the Three-Body problem and other n-Body problems more effectively. This in turn increases the accuracy of our estimations, as we can use more precise data, which, given the chaotic nature of the solar system, is vital. Calculations also become more rapid, meaning we can perform more of them.

The chances of our solar system being perfectly stable forever are extremely slim. Use of the Verlet Method and Fast Multipole Method can provide us with some reasonable comfort that during our lifetimes, we will not crash into the Sun. However, unless a general solution for Three-Body (and n-Body) problem is found, we will never be quite certain of the exact motion of our planet. Perhaps Newton had the right idea – to take comfort in a higher power when faced with that which we cannot predict.