<u>The Isoperimetric Problem: From Bull-hide to Bees</u>

In the hope of escaping her murderous brother, Princess Dido fled her homeland, Pheonicia, in 825 BC. She sailed the Mediterranean with some of her fellow citizens to the northern coast of Africa.

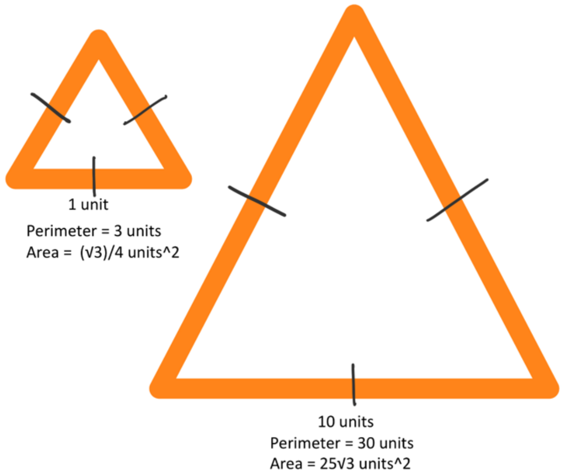
"*Here they bought ground; they used to call it Bysra,*
That being a word for bull's hide; they bought only
What a bull's hide could cover."

Princess Dido's possession of land on the northern coast of Africa was constrained to the area of a "*Bull's hide"* as stated by Virgil. Her impressive intuition led her to cut the bull's hide into thin strips, like what we see when we are introduced to the concept of finding areas between curves, more widely known as integration. But what was even more miraculous, was how she placed these strips in the shape of a semi-circle, whose diameter was the shoreline. On this foundation, the city of Carthage was built.

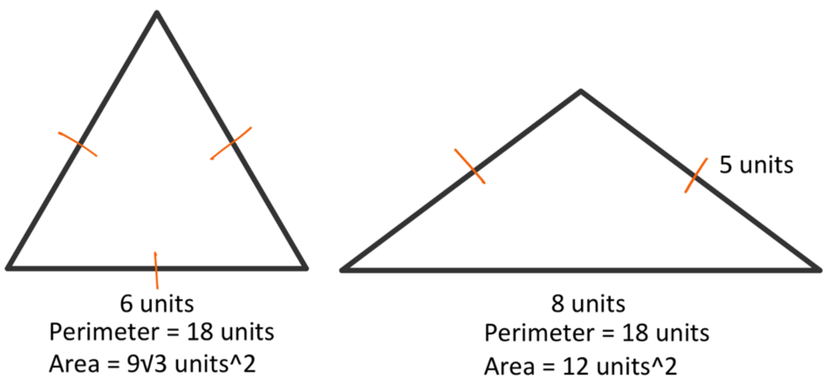
The story of Princess Dido provided mathematicians with the mythical ancestry of a prominent problem: The Isoperimetric Problem ("iso" meaning the same, "perimetric" meaning perimeter or boundary). This problem in mathematics seeks to find the shape that encloses the largest area within a fixed perimeter. Dido reasoned that it was the semi-circle that enclosed the largest area, but mathematicians weren't convinced of this result, perhaps due to Dido being a mythological being and the frequent reference to a bull's hide in her reasoning.

This problem gave Greek geometers a run for their money; even though geometers weren't incentivised by money, they loved the pride that came with proving conjectures. Furthermore, this problem regained prominence two millennia later as a test for the newly emerged calculus.

My initial reaction to this question was how it is possible for 2-dimensional shapes with the same perimeter to have different areas. Proclus and other Greek Geometers matched my confusion when they first heard the problem. They initially believed that the larger the perimeter, the larger the area, which is true in some cases, but not all, as shown below.



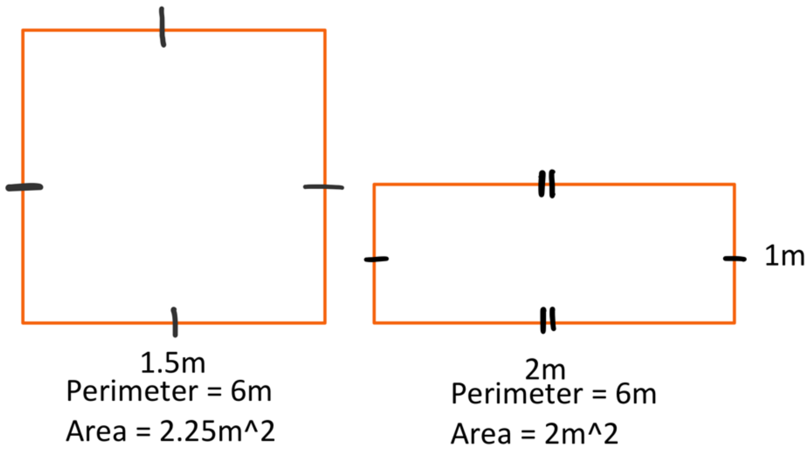
My confusion was short-lived as I drew triangles with perimeter of 18 units. The first triangle I drew was equilateral, three sides of 6 units each. The second triangle I drew was an isosceles triangle with a base of 8 units and legs of 5 units.



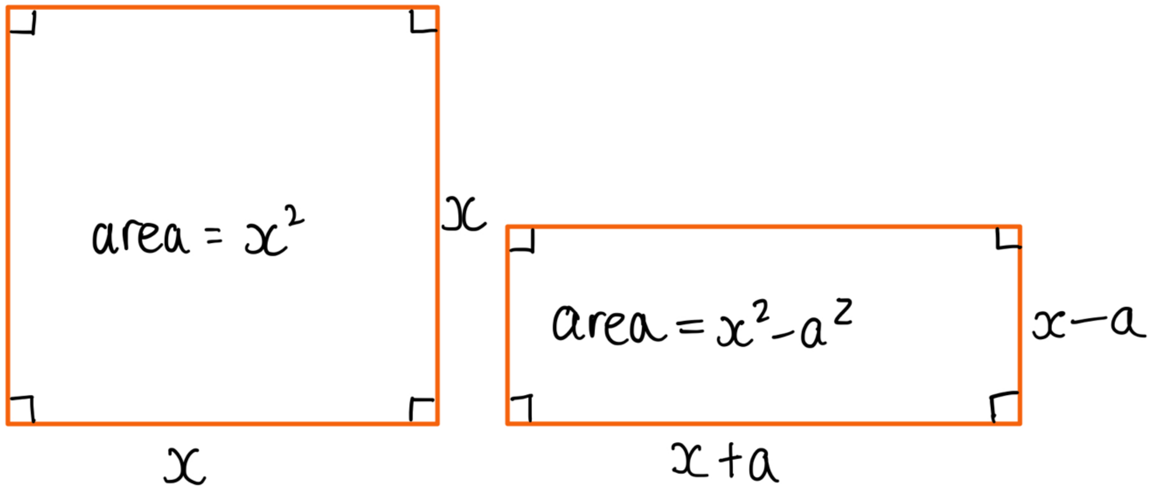
Evidently, they don't have equal areas. Building on this logic, increasing perimeter does not necessarily result in an increase in area, area is dependent on the way you choose to design your shape…but perimeter and area are interrelated somehow.

In fact, it is the equilateral triangle that encloses the greatest area of a fixed perimeter. This can be proven using calculus, and Heron's formula (lovely proof, but way too much number crunching to include here).

Let's move on from triangles to quadrilaterals. Suppose I have 6 metres of string. Should I use it to enclose a square or rectangle to maximise its area? With some experimentation, I realised that no non-square rectangle could reach the area of a square with any given perimeter.



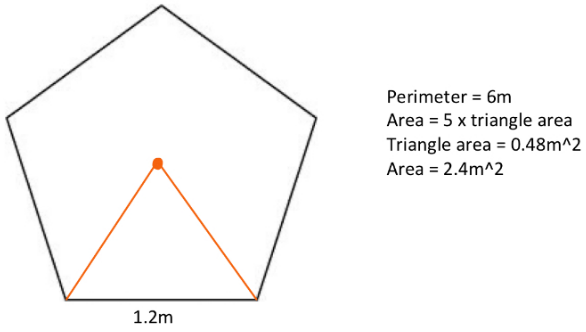
Is there a pattern beginning to emerge? Do all regular polygons enclose the largest area within a fixed perimeter? The space enclosed by an equilateral triangle is greater than any other form of triangle, and a square is more spacious than any other quadrilateral. Luckily, proving that of all rectangles of the same perimeter, it is the square that encloses the greatest perimeter is quite simple (much simpler than proving the same for triangles).



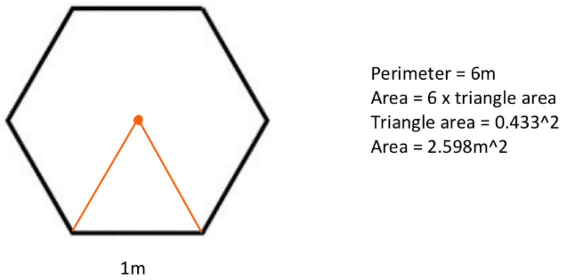
Proof:
Suppose we have a fixed perimeter 4x. From this, we can make a square, with sides of length x.
The area of this square is x^2.
We can also make a rectangle by lengthening the horizontal sides to x+a, and shortening the vertical sides by the same amount (a) giving us a side length, x-a. It follows that the area of this rectangle is (x+a)(x-a) = $x^2 - a^2$.
This is clearly less than x^2. A non-square rectangle of a fixed perimeter encloses an area smaller by a^2 than the square of the same perimeter. ■

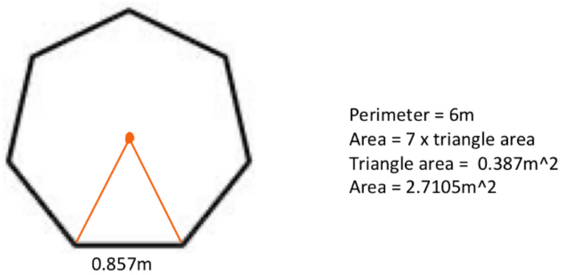
This visual proof is one of many ways in which mathematicians can prove a branch of the isoperimetric problem for quadrilaterals. Of course, this can be proved with our maximising techniques using differential calculus.

Why should we stop here? The Isoperimetric Problem relates all 2-dimensional shapes. What would happen to the area enclosed by our rope should we arrange it in the shape of a pentagon, a hexagon or an n-gon? It is a fact that regular polygons enclose the largest area. Nature just loves symmetry!



NB: the distance between the centre and each vertex must be the same for each polygon. I have made it 1m.





The area of the successive polygons increases with increasing number of sides. However, this is insufficient as a proof for my conjecture [as the number of edges increases, the area of the regular polygon increases]. Should I choose to draw all regular polygons up until a hectogon (100-sides), and find their area, I still have not proved that with increasing sides, area enclosed increases because I have not displayed this for all infinite polygons that exist. The proof that shows this is quite complicated, so I hope to show this intuitively, rather than in a formal proof.

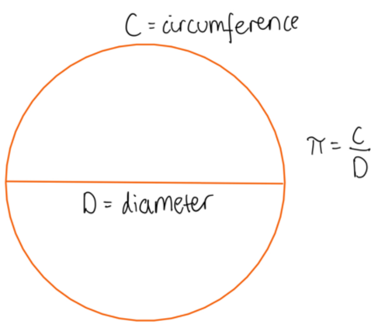
By now, you might be suspecting that the answer to this problem is a Circle. It must be. If area keeps increasing with the number of sides (perimeter remains constant), then what shape encloses the largest area? We don't know what the largest number is, so can we state that the shape that encloses the greatest area is an n-gon where n is the biggest number in the universe?

I hope not.

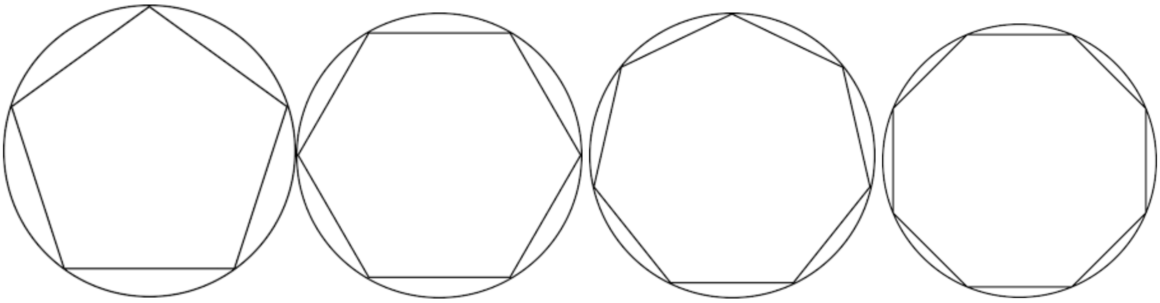
Zenodorus' isoperimetric principle states that for a fixed perimeter, the more sides of a regular polygon, the greater the enclosed area. A circle is the limiting border or outline of a regular polygon with ever-increasing number of sides. It encloses a larger area than any regular n-gon that can be drawn within it.

Fine. But how is a circle the limiting figure of a regular polygon?

We all know pi as the ratio of a circle's circumference to its diameter. Measuring the circumference and diameter using a tape measure comes with huge uncertainty, so mathematicians could not calculate pi using that method. Instead, pi was computed using geometry. A straightedge and a pencil (drawings in sand if you're Archimedes). Mathematicians realised they could not find pi from the circle alone, they had to use straight-edged shapes inscribed in circles and find their area's limit.

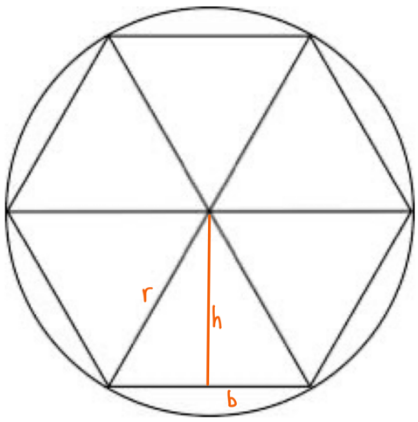


Within a circle, regular polygons can be inscribed. The perimeter of the polygon is an approximation for the circumference of the circle. As the number of edges of the polygon increases, the approximation for the circumference of the circle improves and becomes more accurate. Just like Archimedes used polygons inscribed inside circles to determine the magnitude of pi, the same concept used to get closer to a solution for the Isoperimetric Problem.



You can see that as the number of edges increases, the perimeter gets closer to the circumference of the circle and as a result the area of the polygon will tend to the area of the circle. Archimedes could now attack the computation of pi by finding the area of the polygon inscribed within it, or by finding what the area tends to when the perimeter gets closer and closer to the circumference. We can see clearly from the images above, that as the number of sides of the regular polygon increases, more and more of the circle is filled. What happens when the number of sides of the regular polygon goes to infinity?
It is quite clear that the polygon will gradually fill up the circle. The area of the circle is the limit of the area of the polygon.

We know how to find the area of an n-sided regular polygon. We find the area of the n little triangles the polygon is made of, and multiply this value by n. Each triangle will have apothem (distance between centre of circle to the centre of the sides of the polygon), h, and base, b.



Area (triangle) = $\frac{1}{2}$(base)(height)

$=\frac{1}{2}$bh

Area (n-sided polygon) = n x $\frac{1}{2}$bh

= (nb) x ($\frac{1}{2}h$)

= perimeter x $\frac{1}{2}$h

Therefore,

Area (circle) = lim($\frac{1}{2}h \times\ perimeter$)

lim h = r   lim(perimeter) = C (2πr)

Area (circle) = $\frac{1}{2}r\ \times C$

$= \frac{rC}{2}$

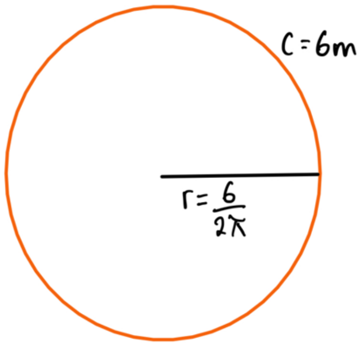
$= \frac{r \times 2\pi r}{2}$

$= \pi r^2$

Pi makes an appearance.
The formula for the area of a circle makes an appearance.

I have just shown that the limit of the area of a polygon is in fact a circle.

To confirm Zenodorus' principle, the area of a circle with circumference 6m is:



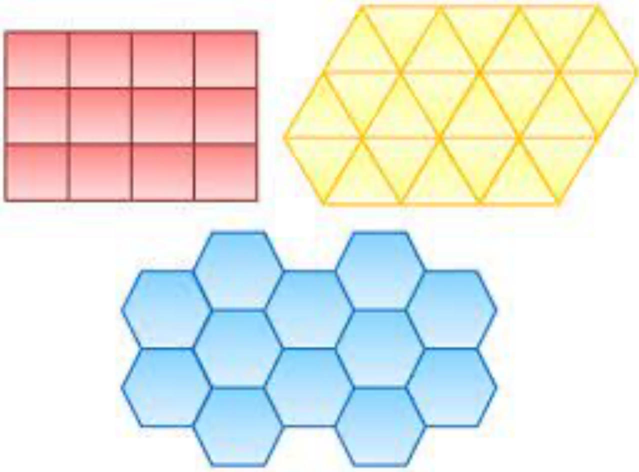
$$\text{Area(circle)} = \left(\frac{6}{2\pi}\right)^2 \times \pi$$
$$= 2.865m^2$$

This is larger than any regular polygon. Try find the area enclosed by a hectogon inscribed within a circle with radius 1. I don't recommend it because it will be long and painful calculation, but the area will not reach the area of the circle, trust me.

So why did Princess Dido arrange the bull-hide strips into a semi-circle? My best guess is that the land she planned on acquiring was on the coast. I'd like to say that had she established a city that wasn't on the coast, somewhere landlocked, she would've gone for a circular border. However, going for a semicircle was still her best option considering her land was along the shore.

Greek mathematicians established that irregular polygons aren't area efficient, but irregular curves slipped their minds. Parabolas and ellipses were not considered when they were discussing solutions to this problem. Luckily for them, the oldest answer remained correct. The circle holds the greatest area out of all 2-D shapes for any given perimeter.

How is this problem present in nature?



Bees are natural geometers. The bees solved the isoperimetric problem before any of us humans. They didn't need a formal proof or any calculations to be satisfied that the shape that can store the most honey and can tessellate is the hexagon. Pappus believed that bees didn't want to waste any honey. They wanted to preserve it all for human consumption, and as a result found the most effective way of doing this.

There are only three regular polygons that tesselate. Triangles, squares, hexagons. Bees realised that hexagons could hold the most honey and can tesselate, thus maximising honey in all ways possible. Bees might not have needed a proof for this, but I do.

Proof:

Suppose we have a regular polygon with n sides, each angle with size α. In this problem, $n \geq 3$ because polygons must have at least 3 sides. From its centre we can draw lines to each vertex of the polygon thus dividing the regular polygon into n identical triangles. We could now calculate the degree total via two different methods.

The triangles in the polygon each contain 180°. The total number of degrees in the whole polygon is therefore n x 180º however this includes the 360° in the of the polygon. The base angles of each triangle is $\frac{\alpha}{2}$ there are 2n of these base angles +360°. Therefore, the total number of degrees in the triangles composing a polygon is:

$$360 + \left(2n \times \frac{\alpha}{2}\right)$$

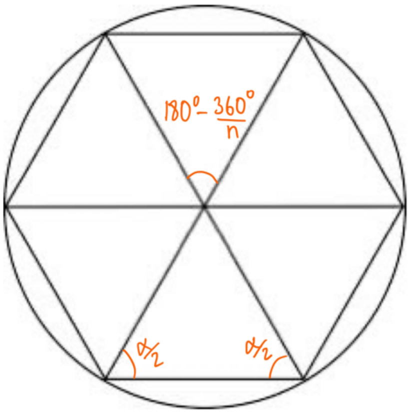
$$= 360 + n\alpha$$

Indeed,

$$360 + n\alpha = 180n$$

$$\alpha = \frac{180n - 360}{n}$$

$$= 180 - \frac{360}{n}$$



A minimum of three polygons must meet at each vertex so that there are no gaps. Let β be the number of polygons that can meet at each vertex.

$$\beta \geq 3$$

So,

$$360^{\circ} = \beta \times \left(180^{\circ} - \frac{360}{n}\right)$$

Upon dividing both sides by 360º, it follows that

$$1 = \beta\left(\frac{1}{2} - \frac{1}{n}\right)$$

Since $\beta \geq 3$,

$$1 \geq 3\left(\frac{1}{2} - \frac{1}{n}\right) = \frac{3}{2} - \frac{3}{n}$$

$$\frac{3}{n} \geq \frac{3}{2} - 1$$

$$n \leq 6$$

Now we can look at all the cases where $3 \leq n \leq 6$.

1. When n = 3, the polygon is a triangle. From our previous equation we know that $\alpha = 60$º. 360/60 = 6. Therefore, can tesselate.
2. When n = 4, we have a square. $\alpha = 90$º. 360/90 = 4. Therefore, can tesselate.
3. When n = 5, we have a pentagon. $\alpha = 108$º. 360/108 is not an integer, so this case must be discarded.
4. When n = 6, we have a hexogon. $\alpha = 120$º. 360/120 = 3. Therefore, can tesselate.

As claimed, the only regular polygons that can tesselate are equilateral triangles, squares, and hexagons. ■
(hexagons holding the largest area for a fixed "radius").

This essay introduces some interesting concepts surrounding the Isoperimetric problem. It highlights the relation between perimeter and area. The isoperimetric problem is one faced by many aspects of nature on a daily, whether it's bees maximising honey storage or humans messing around with string.