Unlikely scenarios can happen quite frequently. That could be bumping into an old friend down at the pub, or knocking a bottle off a table and it landing perfectly on the floor. Being a student in a small town, I constantly run into my teachers over the holidays. I have seen my house master four times, physics teacher three times, the design teacher twice, and countless others. A while ago I bumped into my maths teacher on the streets of town, and she said 'what are the chances', which raised the question: What are the chances of me running into a teacher that I recognize outside of school?

At first I thought the chances were incredibly low, since there's quite a few people that live in town and not a lot of teachers that I recognize. Then, I decided that the chances are higher than I thought since I'm relatively more active in the local ASDA, Costa, and pizza express, locations which are also some teacher's favourites. Futhermore, I see a lot of people on a daily basis, so that also raises the probability to a higher value, since each person I see has the same probability of being my teacher.

But now I wonder, what do I mean when I posed the problem in the first place? Do I want to determine the chance of me bumping into a teacher on a particular day, or over the course of five years of studying here? To bump into a teacher over the course of five years is not that difficult, and most of my classmates have had at least one encounter. However, the chances of me greeting another teacher on a particular day, my birthday perhaps, is much lower. The question is not well defined, and hence doesn't have a unique solution.

It was certainly difficult to come to conclude the question I asked myself didn't have a unique answer. The chances of meeting a teacher should have a solution, but the question posed doesn't tell me what it is I need to determine although it appears to. This then led me onto investigate other problems in probability, and wonder if there are famous problems that appear to be well defined, but turn out to have no unique solution. Hence, I found Bertrand's paradox which is much simpler than the question I asked myself, and therefore more elegant.

Bertrand's Paradox was first posed by French Mathematician Joseph Bertrand in 1889 in his work Calcul des Probabilités. Bertrand was a mathematician who worked in multiple fields such as Number Theory, Differential Geometry and Probability Theory. Fundamentally, it is a probability question that has three different solutions based on different initial approaches. Bertrand posed this paradox to illustrate the importance of having a consistent method in tackling problems as well as a well-defined question, since all three methods he discusses are theoretically and empirically correct in different ways. After this was published, many attempts have been made to try to prove one of the methods gives the true solution, however they have remained unsuccessful up to this day. Hence, it appears that there are at least three solutions to this bizarre problem. The question certainly was not complicated, and its simplicity is what makes it more interesting than its kind.

The Question

Bertrand's Paradox says the following: Consider an equilateral triangle inscribed in a circle. Suppose a chord of the circle is chosen at random. What is the probability that the chord is longer than a side of the triangle?

In order to begin to write a solution, it is necessary to understand each part of the question means to identify exactly what part of the question is not well defined –

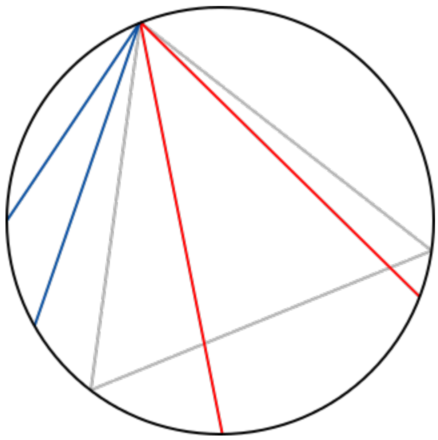
An equilateral triangle is a triangle (three-sided shape) with three equal sides.

For a shape to be inscribed into another is to draw on the inside of with corners just touching but never crossing the side of the outer shape.

A chord is typically defined as a line segment connecting two points on a curve. However, a chord of the circle is a line segment whose endpoints lie on the circumference of the circle. Chosen at random means that in this case, every chord possible has an equal chance of being chosen. In other words, there is a uniform probability of each chord being chosen.

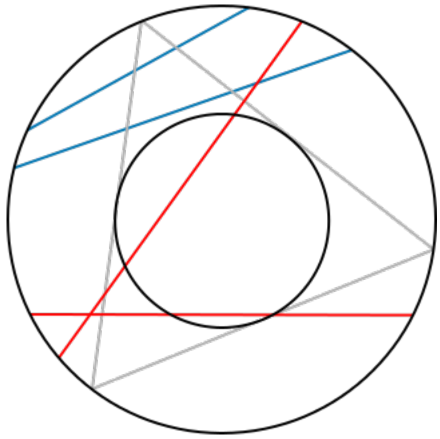
With the definitions out of the way, solutions can now be constructed through different starting points. In this scenario, the different starting points is the different methods of choosing a chord. In the diagrams bellow, the red lines represent chords that are longer than the side of the equilateral triangle, and the blue lines represent chords that are shorter than the side of the equilateral triangle.

Solution 1



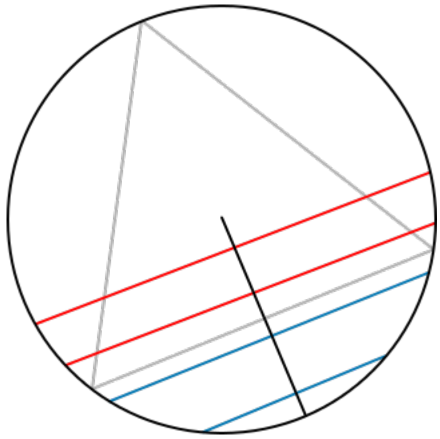
Without loss of generality, it can be said that one end of the chord is chosen at the corner of the triangle. If this is not the case, then the triangle can be rotated such that this is the case since the orientation of the triangle doesn't impact the side length of the triangle. Then, the other end of the chord is chosen at random from the rest of the circumference of the circle. These two endpoints define a chord uniquely and each chord is just as likely to have been chosen since the two points are chosen at random. By observing the second point of the circle it can lie in one of three regions. The region to the left of the first point, the region to the right of the first point and the region opposite to the first point. The regions are separated by the corners of the inscribed triangle. If the second point chosen is to the left or to the right of the first endpoint, then the chord formed is shorter than that of the equilateral triangle. If the second point is chosen in the region opposite, then the chord formed is longer than that of the triangle. Since the triangle inscribed is equilateral, the three regions cover an equal length of the circumference. Hence, the resulting probability that a randomly chosen chord is longer than the side length of the triangle is a third.

Solution 2



Each chord is uniquely defined by its midpoint, meaning for each point within the circle only one chord can be constructed with the chosen point as it's midpoint. Hence, by choosing a random point within the circle, with every point having an equal chance of being chosen, each chord can be chosen with equal probability. Once the midpoint is chosen and the cord is drawn, the triangle can again be rotated such that the closest side is parallel to the chord. If the chord doesn't intersect with the sides of the equilateral triangle than the length of the chord is shorter than the side length of the triangle. If the chord intersects with the sides of the triangle, then the chord is longer than the side lengths of the triangle. Through this method, it can be seen that only if the midpoint of the chord lies in the circle inscribed in the triangle that the length of the chord is longer than the side lengths of the triangle. The radius of the smaller circle is half of that of the outer circle, making the area a quarter of the larger circle. Hence, the probability of a randomly chosen chord being longer than the side length of an inscribed equilateral triangle is a quarter.

Solution 3

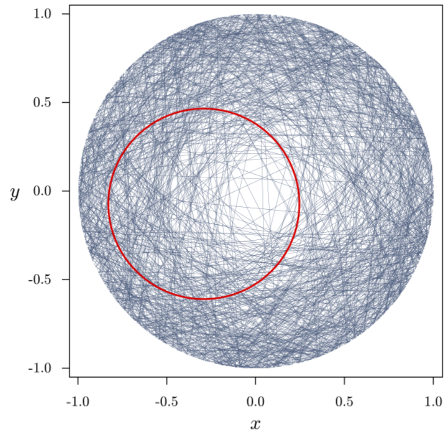


Using the fact that each chord of a circle is uniquely defined by its midpoint, this method first chooses a random radius, then a random point along that radius to define the chord itself. The radius is uniformly chosen first, and the the triangle can be rotated such that one side of it is intersecting and perpendicular to the radius of the circle. Then, a random point is chosen along the radius to be the midpoint of the chord and the chord can be drawn. This gives each chord a uniform chance of being chosen, since the radius is uniformly chosen and the point along the radius is randomly chosen. If the midpoint lies outside the inscribed triangle, then the chord is shorter than the side length of the triangle. If the midpoint lies inside the inscribed triangle, then the chord is longer than the side length of the triangle. Through this method, it can be found that the radius is split in half by the triangle, making the probability of the random midpoint lying within the triangle to be a half, making the probability of a randomly chosen chord being longer than the side length of an inscribed equilateral triangle a half.

The Issue

The truth is that these are not the only three solutions to this problem. Work done by O. K. Bower in 1934 shows that there exists an infinite family of solutions to this problem, which gives further proof that there is no unique solution to this problem. This comes as a surprise since the question itself appears to be a well-defined question. Choose any chord of the circle at random, what are the chances of my chord being greater than a certain length. However, by using different chord selection methods and going through completely logical mathematics, different conclusions are made.

Bertrand used these three solutions to demonstrate that the question simply didn't have a unique solution, although appearing to have one. He argued that only when the method of random selection is specified can the problem have a well-defined solution. Hence, without further information there is no reason for a mathematician to prefer one solution over another. In essence, all calculations give a correct solution. They only differ due to their various starting points.



Attempts have been made to try to resolve the issue. In 1973, a paper call "The Well-Posed Problem" by Edwin Jaynes proposed a unique solution based on the maximum ignorance principle, which means no additional information should be used if it is not given in the question. Jaynes discussed that the problem doesn't identify size or position of the circle, and therefore the solution must remain constant under size and transitional variance. He attempted to demonstrate this by first having a circle of radius two and then a circle of smaller radius, such as one within the large circle. He then performed the methods provided in previous parts with the larger circle, and attempted to see if the chords in circle two would have the same probability. He concluded that only the solution of a half followed this principle, and therefore is the unique solution to Bertrand's paradox. However, in 2015, an article by Alan Drory argued that Jayne's principle can also be achieved with the other solutions by showing each method can be derived using rotational, scaling, and translational invariance, proving that all three methods described by Bertrand followed the principle of maximum ignorance.

There was nothing groundbreaking in terms of difficulty in this paradox, but it is fascinating to see that a simple question which appears to have a well-defined solution can have multiple. Each solution is simple to understand on its own, taking advantage of relatively basic geometry and logic to arrive at each solution. However, when combined together, it is sometimes difficult to understand how a question can have three solutions or more.

This raised an aspect of mathematics I had never considered before. It is important in mathematics to question everything to ensure the solution is rigorous, including the question itself. Sometimes a question simply is not well defined, and therefore acceptable to have multiple solutions. Hence, questioning even the question itself can turn out to be an interesting problem on its own, and definitely a fascinating journey.