A Mathematical Stab in the Dark

The Art of Analytic Continuation

Ewan Hallatt

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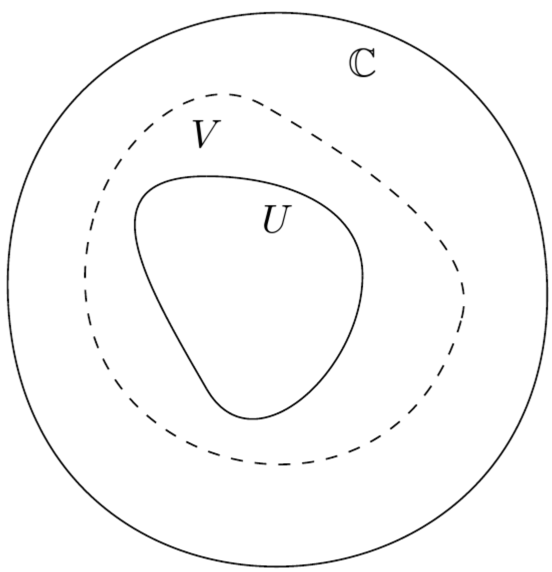


1 Limitations at First

In how many different ways can we arrange 5 things? We start off with 5 things, then have 4 remaining options. Then, after choosing again, we have 3. Then 2. Then 1. The answer to this question comes quite naturally; for every one of the 5 things, we have 4 possible choices from there, then for each of those 4 we have 3 further choices, and so on. Therefore, the answer to our question is $5 \times 4 \times 3 \times 2 \times 1 = 120$. 120 ways of arranging these 5 things. Any competent mathematician will know this is the concept of a *factorial*. We denote the factorial with an exclamation mark, although to me that always suggested the number was shouted when read out loud.

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 2 \times 1$$

Simple enough. The factorial, among other things, is used to tell us how many ways we can arrange n objects.

Let's go a little deeper. How many ways can we arrange 0 things? This is an interesting question, albeit perhaps overly philosophical. But what is mathematics without generalisation? Let's think about this logically. Your common sense would tell you the answer is 0. You have 0 things; how could you possibly arrange 0 things? Whilst this may seem like the obvious answer, there is another way of thinking about this problem. Imagine you have 0 things. You clearly can't rearrange them, so they must (and will always) be in their only possible arrangement. That's one arrangement.

Despite maybe being counter-intuitive at first glance, this is actually the way mathematicians define it. $0! = 1$. This isn't the only reason why, as there are patterns to the factorials that you would want $0!$ to follow:

$$\begin{aligned} 3! &= 3 \times 2! \\ 2! &= 2 \times 1! \\ 1! &= 1 \times 0! \\ \text{so } 0! &:= 1 \end{aligned}$$

("$:=$" means "defined as equal to")

This is one of the many examples of mathematically "stabbing in the dark" when it comes to new ideas and definitions in maths. It mainly boils down to making the new definition fit in with what is already there, and what we already know about how it *should* behave.

Despite the many use cases of the factorial, especially in a field of maths called combinatorics[1], it's only defined for the natural numbers and zero, denoted $\mathbb{N}_0$ (i.e. 0, 1, 2, ...). This is very limiting, since if we ever wanted to evaluate the factorial of a number outside of this domain, for example $1/2$, it would be impossible. Is there even a logical way of doing this?

[1] Combinatorics

2 Why?

All this talk of further generalisation beyond $\mathbb{N}_0$ raises the question: Who cares? Why on earth would we want to evaluate factorials at anything outside of $\mathbb{N}_0$? What would this even mean?

One of the most popular uses of the factorial is known as the *choose* function, denoted[2] $\binom{n}{k}$. It represents the number of ways we can choose k things from n things, without caring about their order, and has the following formula:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

The explanation for this formula can be thought of as follows: Imagine having n things, and wanting to choose k things from them. We have n options at the start, then $n-1$, then $n-2$, then $n-3$, all the way down to $n-k$ - and that's where we stop. In other words, we want all the terms from $n-k$ and onward to cancel. dividing $n!$ by $(n-k)!$ has this effect, and you can convince yourself this is true by expanding them out. But why the $k!$ on the denominator? The choose function does not care about order. That is to say, if we choose the same k objects, but in a different order each time, they are considered the same. We know there are $k!$ ways to arrange k objects, so we can divide by $k!$ to show we don't care about their arrangement.

As it turns out, the choose function is used to expand *binomials*, which are algebraic expressions with 2 terms. You will have definitely encountered these at school, even if you never went on to expand a binomial in your life. There's an exceedingly useful yet daunting formula for this.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Now, you absolutely **DO NOT** need to try and understand this if you don't already. In fact, most teachers don't bother explaining it to students, and perhaps don't even fully understand it themselves, as it may never have been explained to them! All I would like you to take from this is that the choose function, and therefore the factorial, is extremely useful for different reasons in the wider scope of maths.

Let's consider a slightly different case. What if we wanted to expand $\sqrt{x+y}$? This is equivalent to $(x+y)^{1/2}$, which raises the need for new definitions. How do we evaluate factorials at rationals? This is outside of $\mathbb{N}_0$, thus providing a demand for generalisation - a common theme in the mathematical sciences.

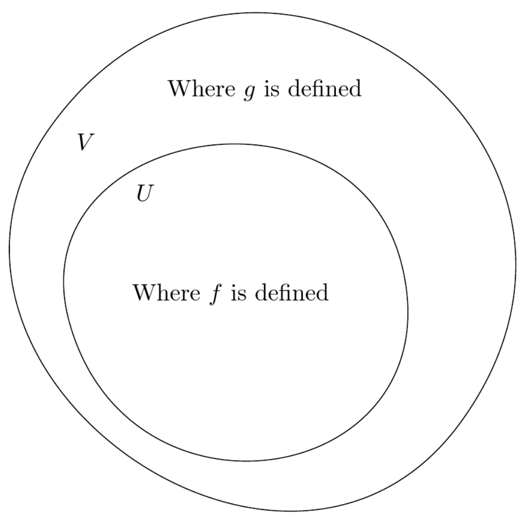
[2]Can also be denoted C_k^n or ${}^nC_k$

3 Extending into the Unknown

There is a logical way to extend the domains of functions like these, called *analytic continuation*. There is a formal definition, although I will explain the concept, so you don't need to pay attention to it unless you know the notation and terminology.

Definition 3.1 (Analytic Continuation)**.** If there exist sets U and V such that $U \subset V \subseteq \mathbb{C}$, and functions $f : U \to \mathbb{C}$ and $g : V \to \mathbb{C}$, then g is the analytic continuation of f iff $f(z) = g(z), \forall z \in U$

Again, you don't need to understand this formally, only the concept. Let's say we have a function f, that is only defined in a domain we'll call U. Let's say there exists another function g in a new domain V which encompasses all of U. If g outputs the same values as f in all of U, then we can say f would be g if we were to extend it to V.

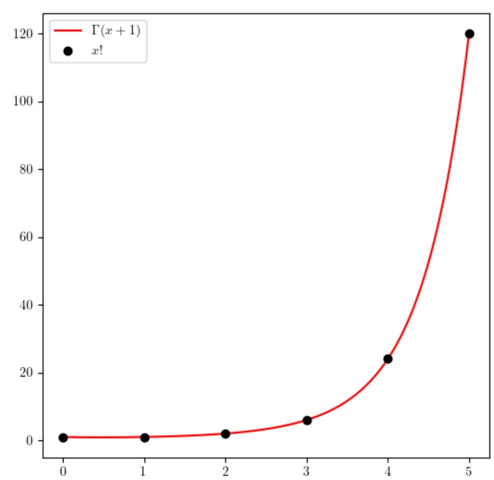


Although this may seem hand-wavy and imprecise, which is atypical for maths, it's the most logical way to extend the domains of functions and get what we need. And, most importantly, it works.

In our case, f is the factorial function, and U is $\mathbb{N}_0$. We need to find a function that is equal to the factorial function for all natural numbers and zero, and also is defined for more than just $\mathbb{N}_0$. Mathematicians spent a great deal of time working on this problem, and managed to find and prove a solution that not only extends the factorial to the real numbers, denoted $\mathbb{R}$ (i.e. 0, 1.5, $\sqrt{2}$, π, $5.\dot{1}3\dot{5}$, ${}^{9}/_{10}, ...$), but also the complex numbers[3]! It's called the Gamma function, and it's defined as follows:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

Woah. Unfortunately for some, not all maths is as simple as the factorial, especially when your trying to stretch and expand it beyond the region where its meaning is well established.



But anyone can appreciate the beauty of it. That we can make an intelligent extension of the factorial function - which is discrete by definition - to something continuous, making it lose all meaningful intuition, yet still making logical sense. And to mathematics, that's all that matters, as much as it may bemuse the human instinct.

[3]You don't need to know what these are, either, but for the curious: Complex Numbers