<u>Why 163 is a really cool number (but -262537412680768000 is even cooler)</u>

Kate Robinson

A common topic of discussion among people who like maths (particularly on the internet) is favourite numbers. Common picks include 73, 23, constants such as π, τ, and e, Kaprekar's constant and the Golden Ratio…. the list is quite genuinely endless (I have neglected to mention infinity). A couple of years ago, there was a project called MegaFavNumbers, where Maths YouTubers made videos about their favourite numbers $>$ 1,000,000. In this essay, I will argue that -262537412680768000 might be one of the most interesting.

Martin Gardner claimed on 1st April 1975 in his "Mathematical Games" column of the Scientific American magazine that the expression $e^{\pi\sqrt{163}}$ had an exact integer value of 262537412640768744[1]. This was predicted by an incredible mathematician called Srinivasa Ramanujan. He was one of the world's greatest mathematicians; growing up in poverty, he was self-taught. Despite his almost complete lack of formal training, he made immense contributions to the field of pure mathematics, namely in analysis, number theory, and infinite series. He correctly calculated this value, ending 743 with a trail of 9s (calculating by hand, he did not determine whether these were recurring).

Gardner was wrong. It is very close to an integer, at 262537412640768743.99999999999925…, but he said that the 9s were recurring (and hence the value was an integer). This is somewhat understandable, given the article was an April Fools and this precision is unachievable by most modern calculators.

Yet what strikes anyone instantly as being odd here is how 3 completely irrational numbers – e, π and $\sqrt{163}$ all combine in such a way to produce a value that is so close to being an integer. Indeed, a similar phenomenon takes place with some other square roots.

$$e^{\pi\sqrt{67}} = 147197952743.99998\ldots$$
$$e^{\pi\sqrt{43}} = 884736743.9998\ldots$$

The square roots are all of Heegner numbers, which were coined by Conway and Guy. They can be determined through a special case of Gauss' class number problem. He was trying to find an exhaustive list of Pythagorean triples. To do this he created a new number system, adding i to all

[1] Gardner, 'Mathematical Games'.

whole numbers. This is expressed mathematically as $\mathbb{Z}[i] = a + bi$, where a and b are whole numbers and $i = \sqrt{-1}$. The new system is complete with its own prime numbers (which vary from those in real numbers). Using this, he was able to factorise the expression $a^2 + b^2$ uniquely to its prime factors of $(a - bi)(a + bi)$.

However changing the imaginary number he added (for example, by adding $\sqrt{-5}$ instead) sometimes made the factorisation non-unique. For instance, 6 can be written as $2 \cdot 3$ or as $(1 - \sqrt{-5})(1 + \sqrt{-5})$; there are two ways to decompose it into a product of primes. As it turned out, not all numbers worked in place of i. There are exactly nine that did: 1, 2, 3, 7, 11, 19, 43, 67 and 163. Gauss conjectured that 163 was the last of the series, which was proved by Kurt Heegner in 1952. There were some flaws in his paper, but the result was proved again by Alan Baker and Harold Stark in 1966, the latter stating that the issues were minor in the original.

But back to our main concern - how can we explain this coincidence? To do this, we need to start by examining an elliptic modular function, defined as follows:

$$j(t) = \frac{1+240\sum_{m>0}\sum_{n>0} m^3 q^{mn}}{q(1-q)^{24}(1-q^2)^{24}(1-q^3)^{24}\ldots} \text{ where } q = e^{2\pi i\tau}$$

This is the simplest function that satisfies $j(t) = j(1 + t) = j(-\frac{1}{t})$. We can expand it as a series to powers of q by writing its Fourier series expansion $e^{2\pi i\tau} = \cos(2\pi\tau) + isin(2\pi\tau)$ as a Laurent series.

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 \ldots$$

Notice how 744 is the constant added to the powers of e we've encountered before, and each of the powers of e has the last 3 digits of 743, which with the series of 9s is incredibly close to 744. However when we add 744 to Gardner's incorrect "exact integer value", we get -262537412680768000. Asides from being -640320 cubed (common knowledge to many, I'm sure), this is also the exact integer value our $j(t)$ function from before spits out when we input $\frac{1+\sqrt{-163}}{2}$.

Using our q-expansion, all of it suddenly makes sense. Substituting in our value of τ, the expansion equals $-e^{\pi\sqrt{163}} + 744 - 196884e^{-\pi\sqrt{163}}$ .... Everything after the first two terms is tiny because the powers of e are so large and negative. Rearranging this, we can hence form a pretty accurate approximation for $e^{\pi\sqrt{d}}$, where d is a Heegner number.

$$e^{\pi\sqrt{d}} \approx j\left(\frac{1+\sqrt{-d}}{2}\right) + 744$$

This is why it's so close to an integer!

As to why I think this should be up there with other favourite numbers (that might be slightly less clunky and more easily expressed), I think the maths speaks for itself. A lot is going on here. 3 irrational numbers came together to produce an almost whole number, a slightly terrifying-looking function is capable of producing whole numbers with the right input, meanwhile, Gauss invented a whole number system resulting in Heegner numbers.

We can look beyond number theory too. In our q expansion, the number 196884 (coefficient of q) shows up because of a mathematical object known as the monster group, discovered by Griess. Of all the small finite groups found in group theory, it is the largest of the 26 'sporadic' ones, with $\sim 8 \cdot 10^{53}$ elements and 196883 dimensions. As it turns out, the monster group is related to the elliptic modular function, hence this coincidence (this is due to a phenomenon known as Monstrous moonshine, where the group unexpectedly connects to various monster functions).

This number also gains us some insight into the mathematical world, where sometimes proofs are wrong and people get away with saying false things in news articles (that aren't even remotely political). Overall, -262537412680768000 is pretty cool, though it's a mouthful to say.

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