ON THE DISTRIBUTION OF PRIME NUMBERS MODULO 180

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ABSTRACT. In this brief paper, we discuss the bias in the distribution of prime numbers modulo 180. Biases in the distribution of primes were first observed for individual primes modulo 4 by Chebyshev in 1853, and for consecutive primes modulo 10 by Oliver and Soundararajan in 2016. Rubinstein and Sarnak were the first to prove unconditionally that a bias occurs for any prime *p* and modulus *q*.[1] We examine the biases in the distribution of both individual and consecutive primes over 48 congruence classes modulo 180, and comment on how these biases relate to and extend the work of Oliver & Soundararajan. We examine implications for twin primes and make several conjectures. We also present preliminary results from our ongoing analysis of the first 63 Mersenne Primes and their associated generating primes using modulo 180.

1. INTRODUCTION

For centuries, mathematicians have successfully used modular arithmetic to detect trends and biases in the distribution of prime numbers. Seminally, in 1853, Chebyshev showed that modulo 4, primes do not occupy the congruence classes 1 and 3 equally often, but instead exhibit a bias towards the congruence class 3.[2] In 1957, Leech showed that modulo 4, the congruence class 1 overtakes congruence class 4 at x = 26,861[3] although the bias towards congruence class 3 persists in the longer run. More recently, in 2016, Oliver and Soundararajan looked at consecutive primes modulo 10 and showed that a prime ending in 1, 3, 7, or 9 is less likely to be followed by another prime ending in the same digit.[4]

It is known that primes must satisfy the following properties:

- **Last Digit** (Primes greater than 5 are congruent to 1, 3, 7, or 9 modulo 10)
- **N modulo 6** (Primes greater than 3 are congruent to 1 or 5 modulo 6 implying the form 6k+/-1)
- **K_N modulo 10** (Primes are of the form 6k +/-1, and their k-values are congruent to 0-9 modulo 10)
- **Digital Root** (Primes other than 3 have a digital root of 1, 4, 7, 2, 5, or 8)

We tabled the first 360 natural numbers according to these four properties and observed that there is a unique combination of values which repeats every 180 numbers. For example, the values for 7 are the same as 187 (it's last digit is 7, it is congruent to 1 mod 6 and of the form 6k+1, its k-value is congruent to 1 mod 10, and its digital root is 7). This is due to the fact that 7 and 187 fall into the same congruence class modulo 180. We then removed any multiples of 2, 3, and 5, and it emerged that primes greater than 5 are distributed among exactly 48 congruence classes modulo 180:

[1] Michael Rubinstein and Peter Sarnak. Chebyshev's bias. *Experimental Mathematics*, 3(3):173 – 197, 1994.
[2] Andrew Granville and Greg Martin. Prime number races, 2004.
[3] John Leech. Note on the distribution of prime numbers. *Journal of The London Mathematical Society-second Series*, pages 56–58, 1957.
[4] Robert Oliver and Kannan Soundararajan. Unexpected Biases in the Distribution of Consecutive Primes 1603.03720.pdf (arxiv.org)

N mod 180	Last Digit	N mod 6	k_N mod 10	Digital Root
1	1	1	0	1
7	7	1	1	7
11	1	5	2	2
13	3	1	2	4
17	7	5	3	8
19	9	1	3	1
23	3	5	4	5
29	9	5	5	2
31	1	1	5	4
37	7	1	6	1
41	1	5	7	5
43	3	1	7	7
47	7	5	8	2
49	9	1	8	4
53	3	5	9	8
59	9	5	0	5
61	1	1	0	7
67	7	1	1	4
71	1	5	2	8
73	3	1	2	1
77	7	5	3	5
79	9	1	3	7
83	3	5	4	2
89	9	5	5	8
91	1	1	5	1
97	7	1	6	7
101	1	5	7	2
103	3	1	7	4
107	7	5	8	8
109	9	1	8	1
113	3	5	9	5
119	9	5	0	2
121	1	1	0	4
127	7	1	1	1
131	1	5	2	5
133	3	1	2	7
137	7	5	3	2
139	9	1	3	4
143	3	5	4	8
149	9	5	5	5
151	1	1	5	7
157	7	1	6	4
161	1	5	7	8
163	3	1	7	1
167	7	5	8	5
169	9	1	8	7
173	3	5	9	2
179	9	5	0	8

Table 1. The 48 congruence classes for primes (greater than 5) modulo 180

Modulo 10, there are 4 congruence classes for primes greater than 5 (1, 3, 7, and 9) which are defined by and therefore associated with only one key property of primes, namely their last digit. In contrast, using the 48 congruence classes of modulo 180, one obtains multiple properties of any prime under consideration simultaneously. For example, any number which is congruent to 49 modulo 180 has the same properties as 49 in the table (it's last digit is 9, it is congruent to 1 mod 6 and of the form 6k+1, it's k-value is congruent to 8 mod 10, and its digital root is 4). We therefore used modulo 180 in our analysis of the distribution of primes because of this utility.

2. RESULTS AND ANALYSIS

Distribution of Individual Primes Modulo 180

Oliver and Soundararajan showed that up to the first 5.8 million primes greater than 5, slightly more primes end in 3 (1,450,185 or 25.003%) or 7 (1,450,153 or 25.003%) than in 9 (1,449,836 or 24.997%) or 1 (1,449,824 or 24.997%).[5]

Their use of modulo 10 (4 congruence classes) and our use of modulo 180 (48 congruence classes) did not hinder us from direct comparison of results due to the fact that our 48 congruence classes can be easily mapped back to their 4 original congruence classes (12 congruence classes modulo 180 for each of the original 4 congruence classes modulo 10).

We observed the following for the first 5,800,001 primes greater than 5:

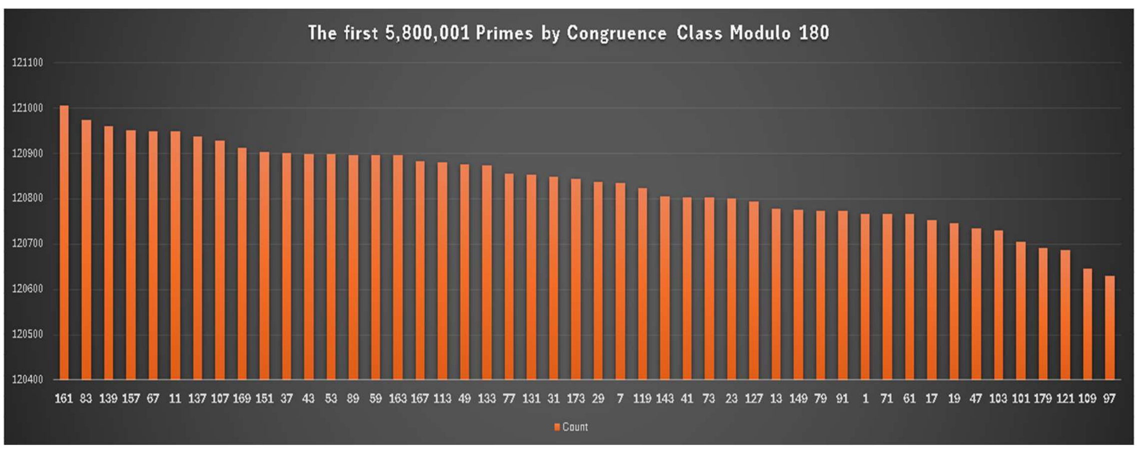

Figure 1. The first 5,800,001 primes (greater than 5) by congruence class modulo 180

Observation 1: Up to the first 5,800,001 primes greater than 5, most primes are in the congruence class 161 which does not end in 3 or 7 (121,005 or 2.09%), seemingly disagreeing with Oliver and Soundararajan's findings.

Observation 2: Up to the first 5,800,001 primes greater than 5, the least primes are in the congruence class 97 (120,631 or 2.08%) which ends in 7, also contradicting Oliver and Soundararajan's findings.

Collectively, our observations show that modulo 10, any primes which end in the same last digit are always in the same congruence class, but modulo 180, those same primes may have different congruence classes and distributions.

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[5] *The Last Digit of Prime Numbers – Numberphile*
available at: https://www.youtube.com/watch?v=YVvfY_lFUZ8

Distribution of Consecutive Primes Modulo 180

Oliver and Soundararajan showed that up to the first billion primes greater than 5, a prime ending in 1, 3, 7, or 9 is less frequently followed by another prime with the same last digit.

Again, their use of modulo 10 (4 congruence classes) and our use of modulo 180 (48 congruence classes) did not hinder us from direct comparison of results due to the fact that our 48 congruence classes can be easily mapped back to their 4 original congruence classes (12 congruence classes modulo 180 for each of the original 4 congruence classes modulo 10).

We observed the following for the first 1,000,000,001 primes greater than 5:

Observation 1: A prime in a congruence class with a particular last digit was less frequently followed by another prime in a congruence class with the same last digit, supporting Oliver and Soundararajan's findings.

Observation 2: A prime in any particular congruence class was never followed by another prime in the same congruence class, supporting Oliver and Soundararajan's findings.

Observation 3: A prime in any particular congruence class was most frequently followed by another prime in the next congruence class.

Oliver and Soundararajan also showed that up to the first billion primes greater than 5, a prime ending in 1 is most frequently followed by another prime ending in 3 or 7 with 30% probability for each.

Table 1 (on Page 2) shows that each of the 12 congruence classes modulo 180 which end in 1 (1, 11, 31, 41, 61, 71, 91, 101, 121, 131, 151, and 161) is followed by a congruence class which ends in 3 or 7 (7, 13, 37, 43, 67, 73, 97, 103, 127, 133, 157, and 163), supporting Oliver and Soundararajan's findings.

Additionally, Observation 3 above shows that a prime in any congruence class was most frequently followed by another prime in the next congruence class specifically, not just a congruence class which ends in a 3 or 7, also supporting Oliver and Soundararajan's findings.

Distribution of Twin Primes Modulo 180

Twin primes are consecutive primes which are two apart (which is the smallest possible gap between primes greater than 2).

Conjecture 3 says that though not always, a prime in any particular congruence class is most frequently followed by another prime in the next congruence class which is exactly the condition that promotes the generation of twin primes. If this holds in the limit, it means that twin primes occur infinitely often, even though they become rarer over time.

From Table 1, it can be seen that there are 18 pairs of consecutive congruence classes that twin primes would have to occupy: (179, 1), (11, 13), (17, 19), (29, 31), (41, 43), (47, 49), (59, 61), (71, 73), (77, 79), (89, 91), (101, 103), (107, 109), (119, 121), (131, 133), (137, 139), (149, 151), (161, 163), (167, 169)

3. CONCLUSIONS

We conclude that the use of modulo 180 proved fruitful in illuminating an underlying structure and natural rhythm of the natural numbers, allowing us to comment on the distribution of prime numbers and on the frequency of twin-primes. Our results show that for each of the 4 original congruence classes modulo 10, its 12 corresponding congruence classes modulo 180 conspire together to produce the biases identified by Oliver and Soundararajan in the distribution of consecutive primes modulo 10. However, we also show that when the 12 congruence classes are analysed individually, a different picture emerges.

We are now led to the following conjectures (modulo 180) for all primes greater than 5:

Conjecture 1: A prime is one congruence class is never followed by another prime in the same congruence class.

Conjecture 2: A prime in one congruence class is least frequently followed by another prime with the same last digit but in a different congruence class.

Conjecture 3: A prime in one congruence class is most frequently followed by another prime in the next congruence class.

Conjecture 4: Primes are distributed among all 48 congruence classes infinitely often.

Conjecture 5: The congruence class with the largest density of primes changes infinitely often.

Conjecture 6: The congruence class with the smallest density of primes changes infinitely often.

4. ONGOING RESEARCH

We are currently analysing the distribution of Mersenne Primes, their generating primes, their 2^(p-1) counterparts, and their associated perfect numbers using the 48 congruence classes modulo 180. We have found that excluding 2 (the first Mersenne Prime), the 2^{nd} to 8^{th} Mersenne Primes are in congruence classes 7, 31, 91, and 127. We also found that the generating primes for the 63 known Mersenne Primes are distributed among the 48 congruence classes as follows:

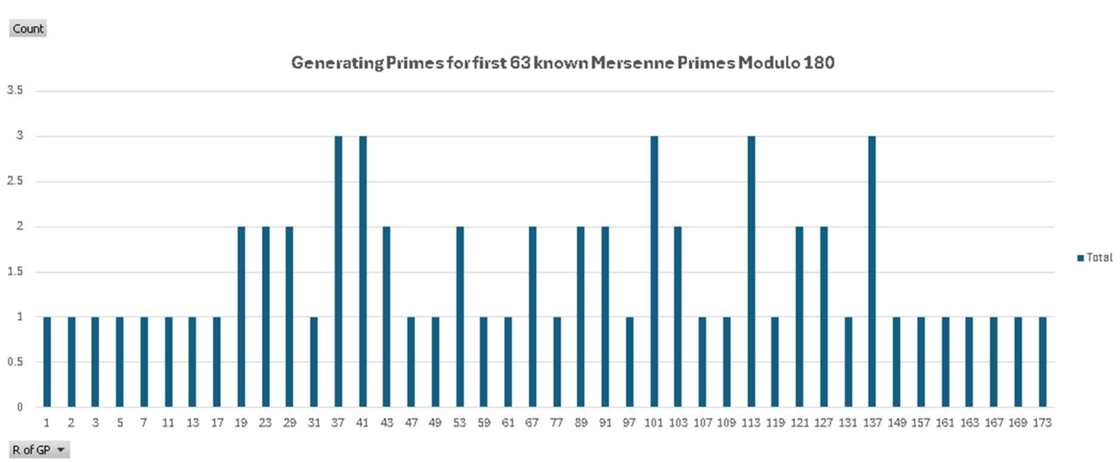

Figure 2. The distribution of the Generating Primes for the first 63 Mersenne Primes modulo 180