A Deep-dive into the Congruency and Uniqueness of Polygons

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Introduction

Most of us were introduced to polygons in school, and were taught how to class them and set them apart (square, equilateral triangle, rhombus, etc.), yet when it came to congruency, we were only taught the theorems for triangles (SSS, SAS, etc.). Here, I hope, with this paper to establish a deeper understanding and set a foundation for congruency theorems for all polygons, using simple and intuitive geometric proofs. And to get a feel visually of how can a polygon be *unique*.

Importance of Researching the Congruency and Uniqueness of Shapes

Understanding and analyzing the properties and characteristics of shapes contribute to the development of numerous practical applications and theoretical advancements.

Mathematical Foundations

Researching the congruency of shapes forms the basis of geometric principles and theorems. By studying congruent shapes, mathematicians establish fundamental concepts related to symmetry, transformations, and geometric relationships. This knowledge serves as a foundation for advanced mathematical topics, such as trigonometry, calculus, and differential geometry.

Practical Applications

The knowledge gained from researching shape congruency is highly applicable in various real-world scenarios. In architecture and engineering, for example, understanding congruent shapes helps in designing structures that are stable, aesthetically pleasing, and efficient. The congruency of shapes is also crucial in fields like computer vision, image processing, and pattern recognition, enabling advancements in object detection, image matching, and augmented reality.

Uniqueness and Identification

Investigating the uniqueness of shapes provides valuable insights into their identification and classification. By studying the distinct properties and characteristics of shapes, researchers can develop algorithms and techniques for shape recognition and classification tasks. This has practical applications in fields such as biometrics, where shape uniqueness is utilized for fingerprint or iris identification.

Visual Representation and Communication

Shapes play a vital role in visual representation and communication. Researching their congruency and uniqueness enhances our ability to accurately convey information through visual means. Whether it is in data visualization, graphic design, or educational materials, understanding and utilizing congruent and unique shapes improve the clarity and effectiveness of visual communication.

Literature Review: Congruency of Shapes

Several research studies have focused on investigating the properties and applications of congruent shapes.

One research paper by Daniel B. Hirschhorn (1990) discusses a triangle congruence theorem that, although true, isn't mentioned in any of the high school textbooks. The study also gives examples of complex problems that can be solved simply using this congruency theorem.

A study by Johnson and Smith (2010) explored the congruency of shapes in the context of geometric transformations. They examined various transformations, such as translations, rotations, and reflections, and their effects on congruent shapes. The researchers found that these transformations preserve congruency, providing insights into the fundamental principles of shape congruency.

Another research paper by Lee et al. (2015) examined the congruency of shapes in the field of computer vision. They developed an algorithm that could determine the congruency of shapes in digital images, enabling applications in object recognition and image analysis. The study demonstrated the practical implications of understanding and detecting congruent shapes in real-world scenarios.

Although previous research has made significant contributions to the understanding of congruency in shapes, there are still knowledge gaps that warrant further investigation.

Overall, the literature on the congruency of shapes provides valuable insights into the fundamental principles, properties, and applications of congruent shapes. By building upon the existing research, future studies can deepen our understanding of shape congruency and its broader implications in various domains.

Research Goals and Objectives

The research aims to experiment with the conditions for the congruency and uniqueness of polygons. The following are the specific goals and objectives of the study:

1. Investigate the effects of different conditions on the congruency of polygons:
   - Explore the impact of side lengths on polygon congruency.
   - Analyze the role of angle measures in determining polygon congruency.
2. Analyze the uniqueness of polygons under varying conditions:
   - Examine the effects of side length ratios on triangle congruency.
   - Investigate the relationship between polygon uniqueness and the number of sides.
3. Develop mathematical models and equations to determine polygon congruency and uniqueness accounting for different conditions affecting it.

Methodology

Developing an Intuition

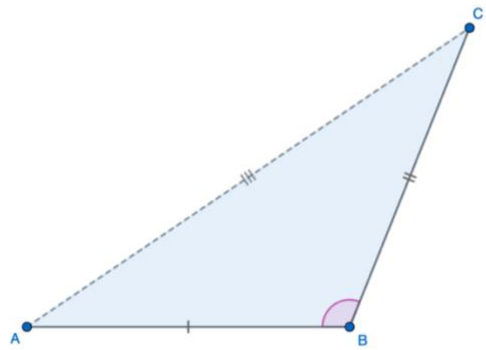
Most of us have been taught in school that to prove two triangles congruent, we must be given certain information about them, such as two corresponding sides being equal in length, or two corresponding angles being equal in measure.

Several criteria can be used to determine the congruency of triangles. These criteria include side-side-side (SSS), side-angle-side (SAS), side-angle-angle (SAA), and hypotenuse-leg (HL) congruence.

Each criterion specifies a particular combination of sides and angles that must be equal for the triangles to be congruent. Let's prove them with basic geometry!

Using the SAS case as a starter, we see that there could only be one way to complete the triangle by joining points A and C, given these conditions. So basically, to prove congruency, it's the same as asking:

"Given these restrictions or conditions, do they restrict our shape to a specific form, a unique form, or can there be multiple shapes that satisfy them"

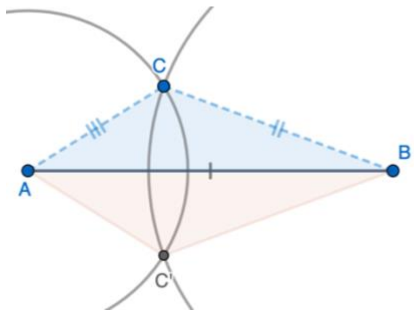


To model these conditions and analyze them geometrically, we will:

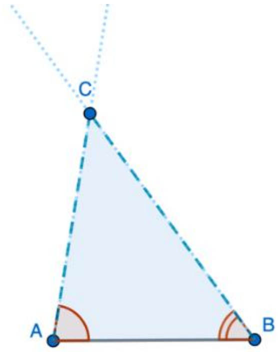
- Label each side and angle with a number from left to right, for example, S_1, A_1, S_2, which will be helpful later on when working with quadrilaterals and polygons in general.
- Use the circumference of a circle to represent a side with an unknown angle, since every point on it will be at a distance "S" from the vertex.
- Use a ray as an angle with an unknown side, since restricting the ray anywhere would represent a side at that given angle "A" to the adjacent side.
- Consider an intersection between constructions as a point that satisfies both conditions of these constructions.

We can see these tools in action in the following diagrams of the cases SSS, SAA, or ASA (both are the same in the case of a triangle) and HL.

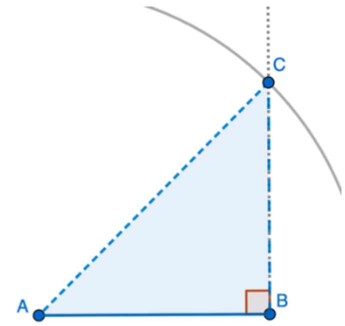
Even though in the case of SSS it appears to have two ways to draw the triangle due to the two intersection points, they are just mere reflections of one another, and hence the same shape. And it is clear from the figures that in the SAA and HL case there can only be one way to draw them given these conditions, due to one intersection point.



S.S.S.



S.A.A.

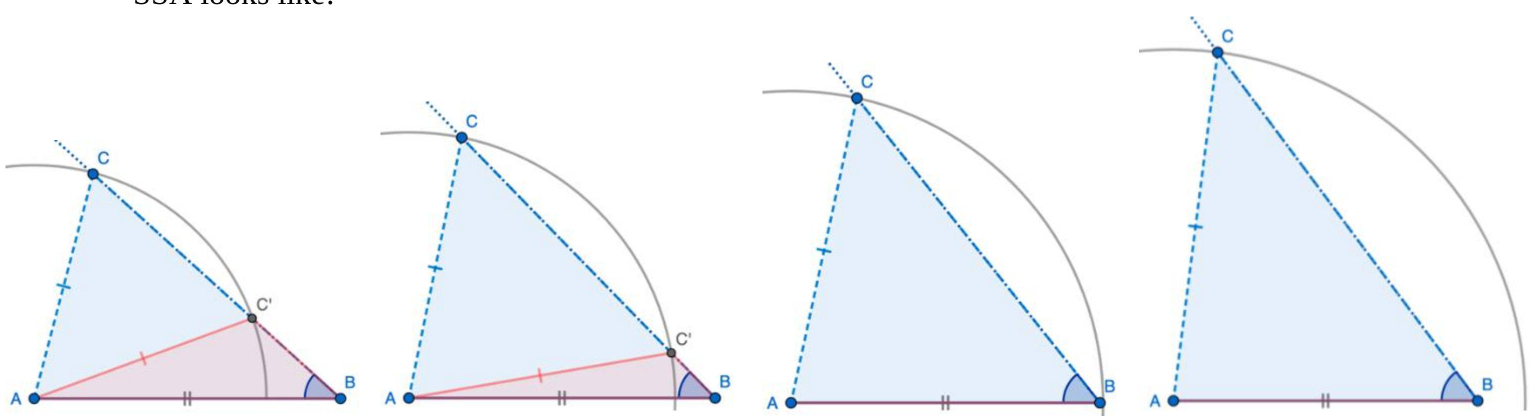


H.L.

The more general case of SSA *

The cases SSS and SAA made sense to me since they were easily provable and quite general, but what I didn't fully get was why the case HL existed.

Yes, we have just proved it true, yet why is it so specific? Why is it only applicable in the case of a right-angled triangle? Why not have a more general condition as SSA? Since technically HL means that you are given two sides and a right angle attached to one side. Well, let's see what SSA looks like!



Although the SSA case seems intuitively valid, it is important to note that it does not guarantee congruence in all cases. There are situations where two triangles with the same SSA proportions and angle do not necessarily have congruent corresponding sides or angles.

In the figures above, in the first case where $S_1 < S_2$. We can see clearly that the ray $\overrightarrow{BC}$ intersects the circle centered at A twice, producing two possible shapes $\Delta ABC \ and \ \Delta ABC'$ which satisfy our initial givens. Hence, we still need more information to prove congruency and uniqueness.

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* HIRSCHHORN, D. B. (1990). Why Is the SsA Triangle-Congruence Theorem Not Included in Textbooks? *The Mathematics Teacher*, *83*(5), 358–361. http://www.jstor.org/stable/27966706

Yet something interesting happens, as S_1 increases and at some point, equals S_2, as seen in the third figure, the second intersection point C' is at point B and therefore leaves us with a single unique triangle ABC, and as S_1 exceeds S_2 in length, point B is *inside* circle A, which implies that any ray coming off B will only intersect the circle *once*—proving its uniqueness!

So, we can deduce that for the case of S_1S_2A , congruency is guaranteed for $S_1 \geq S_2$. In other words, the shape is unique under the condition that the side with two unknown angles is greater than or equal to the side with a known angle.

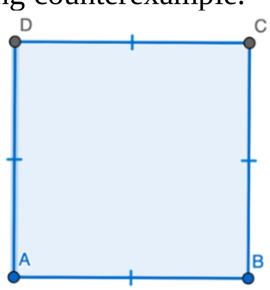
Returning to the case of HL, this now makes perfect sense as the hypotenuse (S_1 or the side with two unknown angles) will always be greater than the leg of the right-angled triangle (S_2 or the side attached to the right angle). In other terms, HL is a special case of SSA.

Congruency in Quadrilaterals

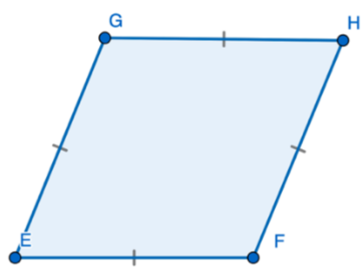
Moving on to quadrilaterals, which are geometric shapes with four straight sides and four angles. Understanding the concept of congruency in quadrilaterals is essential for analyzing and comparing different shapes and their properties.

So, what are exactly the cases under which a quadrilateral will be unique? Let's explore the different possibilities and test to find out ourselves!

Starting with the first case that comes to mind, SSSS, this can easily be proven insufficient by the following counterexample.



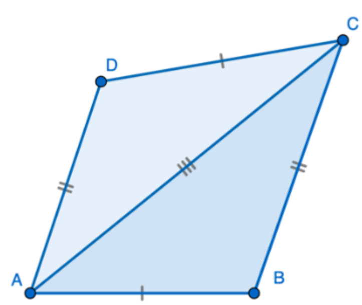
Square



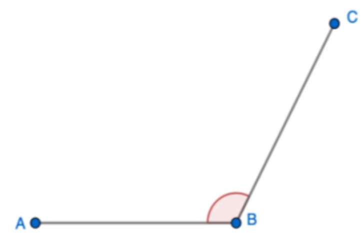
Rhombus

As we can see even though these two shapes have congruent sides, it doesn't mean at all total congruency. But if we add a condition that specifies a diagonal's length, they will be congruent.

That's because the diagonal divides the quadrilateral into two triangles which both are unique due to the SSS case. So overall the quadrilateral will be unique and congruent to any other quadrilateral with the same given

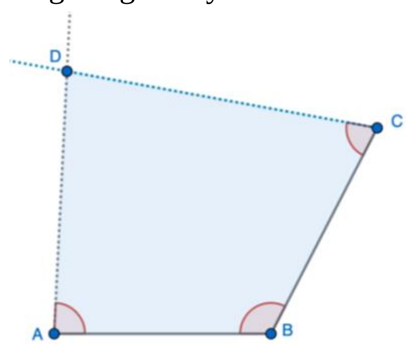


But for consistency's sake, let's keep the givens restricted to sides and angles, with no diagonals. We can still use the trick of dividing the polygon into two triangles but instead of using the case of SSS and needing a diagonal as an input, we can use SAS as we can be given the angle between the two sides, SAA and SSA will not be viable as the angles that will be given do not represent the angles of the two smaller triangles.

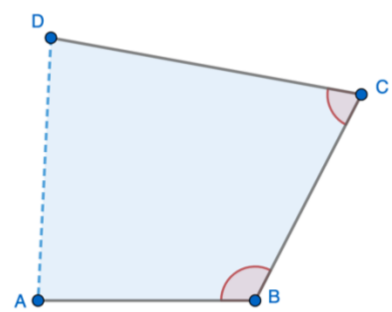


Well, we can start by laying the foundation which is SAS, and start by trying to expand on it to create a fourth point with a unique position.

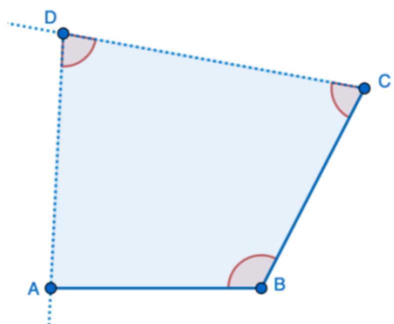
We can create an intersection by adding two angles at points A and C, as in the following figure, creating a unique shape and proving congruency case ASASA.



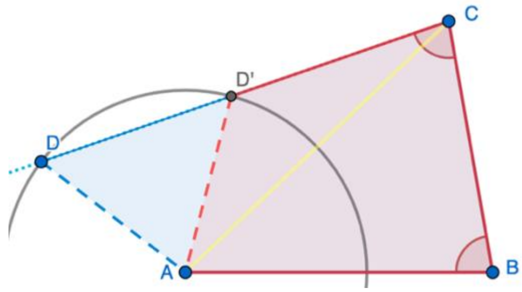
Or we can fix the position of the fourth point by using an angle and a side in series, as in the following figure, which also creates a unique shape and proves the SASAS congruency case.

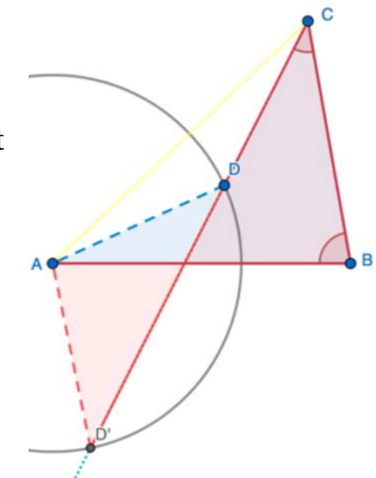


Also, we can use two consecutive angles proving SASAA true, which is different in this case from ASASA. You can imagine it as sliding the ray $\overrightarrow{DA}$ along the ray $\overrightarrow{CD}$ until it clicks with A.



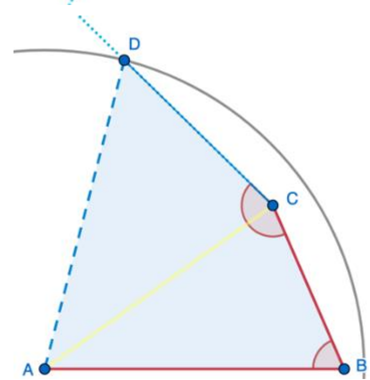
For the case SSASA, we will encounter some problems as the SSA case, and hence we will apply some extra conditions to ensure congruency, take a look at the following figure.





We get the same problem being that the ray produced by the angle intersects the circle twice producing two possible quadrilaterals $ABCD$ and $ABCD'$. Note that this is only the case when the quadrilateral is convex, since we can't determine its convexity given this limited information, we will just assume that it is convex.

The reason for that is presented in the figure to the right. When point D has an angle greater than 180° (or under the yellow line visually), even if the ray intersects the circle twice, the second intersection creates a quadrilateral where two of its sides intersect (as seen in quad. $ABCD'$), which is called a self-crossing quadrilateral and, in this paper, we will only consider *simple quadrilaterals*, which are defined by being a non-self-crossing quadrilateral. So, after rejecting the second intersection there will only be one intersection at point D.



Now returning to convex cases, in order to have only one intersection a condition must be satisfied which is that $AD \geq AC$, that is in order to make point C inside of the circle centered at A, so that there can only be one intersection if the ray $\overrightarrow{CD}$ and the circle's circumference.

As seen in the figure to the right. Since AC is not given, we can formulate it using the law of cosines, which states that in any given triangle,

$$c = \sqrt{a^2 + b^2 - 2ab\cos\theta}$$

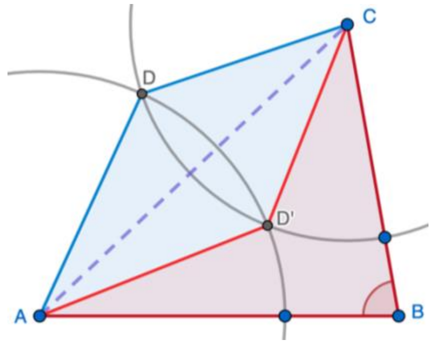
Where a, b and c are the sides and θ is the angle between a and b. Using it in ΔABC yields,

$$AC = \sqrt{AB^2 + BC^2 - 2(AB)(BC)\cos(\angle ABC)}$$

So, writing the condition for which SSASA is true using the terminology of this case,

$$S_1 \geq \sqrt{{S_2}^2 + {S_3}^2 - 2(S_2)(S_3)\cos A_1}$$

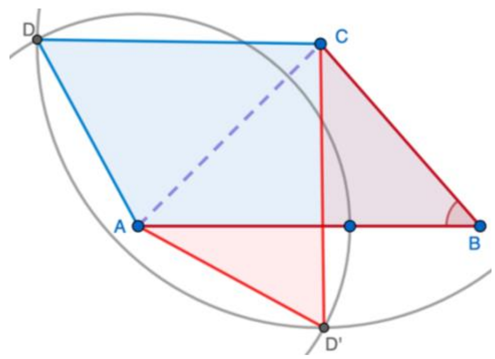
For the final case for quadrilaterals, it is SSASS, as shown in the following figure.



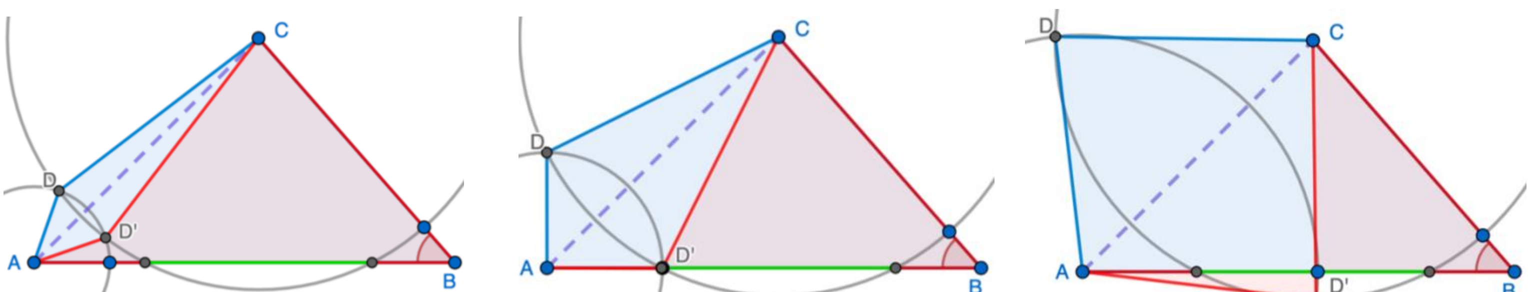
Even though it would be logical to say it will always be congruent, since it is an SAS triangle attached to an SSS triangle, but then we will be dismissing the reflection of the SSS triangle on the line AC, which creates a concave polygon $ABCD'$.

If there was a condition saying that the quadrilateral was either strictly convex or strictly concave, then this case would always be true. But let's see if any conditions can be applied as in the cases of SSA and SSASA, which guarantee uniqueness and congruency in either case.

As you can see in the figure to the right, there will be some cases where again, the second intersection will not result in a simple quadrilateral, which leaves us with a single unique shape, so how can we mathematically describe these situations?



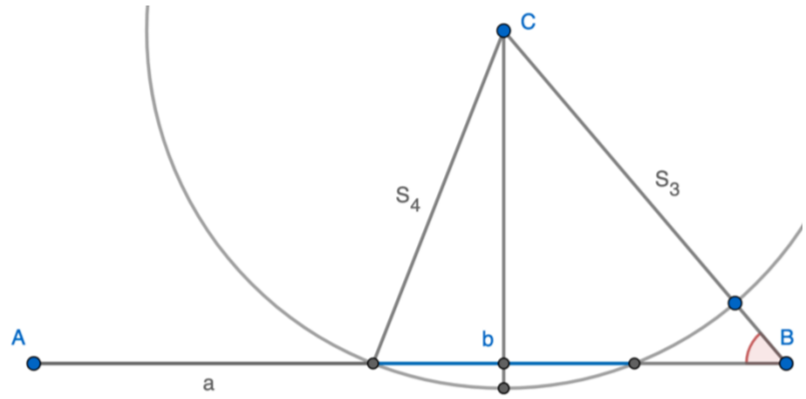
Well after some geometric analysis, we can see that they occur when the radius of one of the circles is in between the two intersection points of the other circle with the opposing side.



Or as seen here when the point of the intersection of the circle centered at A and the line AB is in the green line drawn out by the intersection of the other circle with the same line.

We can imagine it as holding a compass at A, and then drawing an arc that crosses AB, for it to meet the arc of the other circle centered at C on the other side of AB, this arc must start before where the compass arc crossed AB and end after it.

We can formulate it using basic geometry, we can use the following model of one of the sides.



We want the length S_1 to be between a and $a+b$, let's define a, b and the condition itself.

$$a = S_2 - \frac{b}{2} - S_3 \cos A$$

$$\left(\frac{b}{2}\right)^2 = {S_4}^2 - {S_3}^2 (\sin A)^2$$

$$b = \sqrt{4(S_4)^2 - 4(S_3)^2(\sin A)^2}$$

$$a \leq S_1 \leq a + b$$

$$S_2 - \frac{b}{2} - S_3 \cos A \leq S_1 \leq S_2 + \frac{b}{2} - S_3 \cos A$$

$$-\frac{b}{2} \leq S_1 - S_2 + S_3 \cos A \leq \frac{b}{2}$$

$$|S_1 - S_2 + S_3 \cos A| \leq \frac{b}{2}$$

$$b - 2|S_1 - S_2 + S_3 \cos A| \geq 0$$

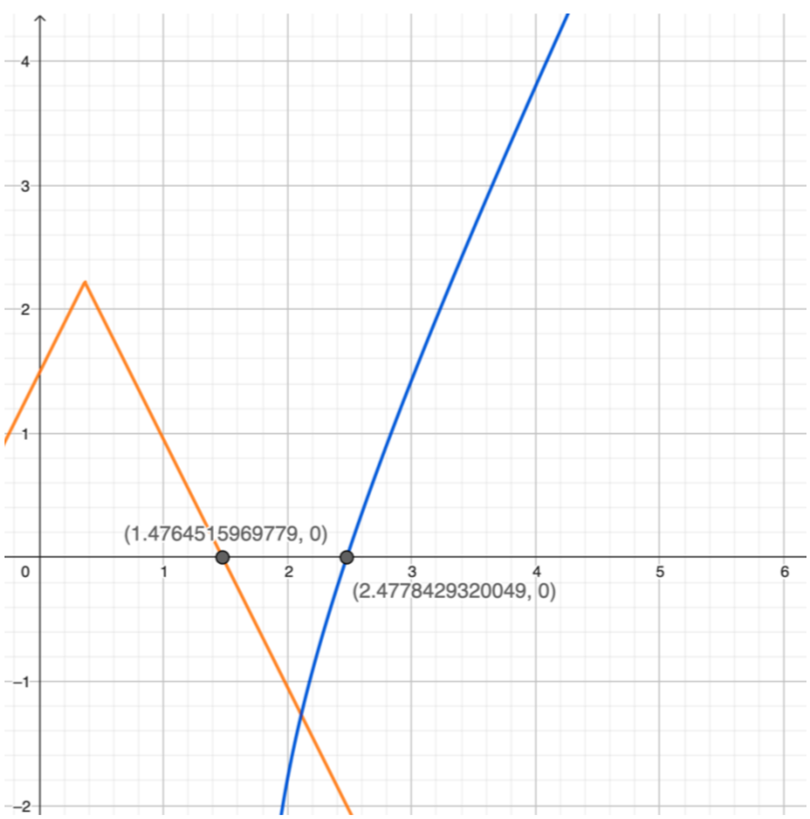
$$\sqrt{4(S_4)^2 - 4(S_3)^2(\sin A)^2} - 2|S_1 - S_2 + S_3 \cos A| \geq 0$$

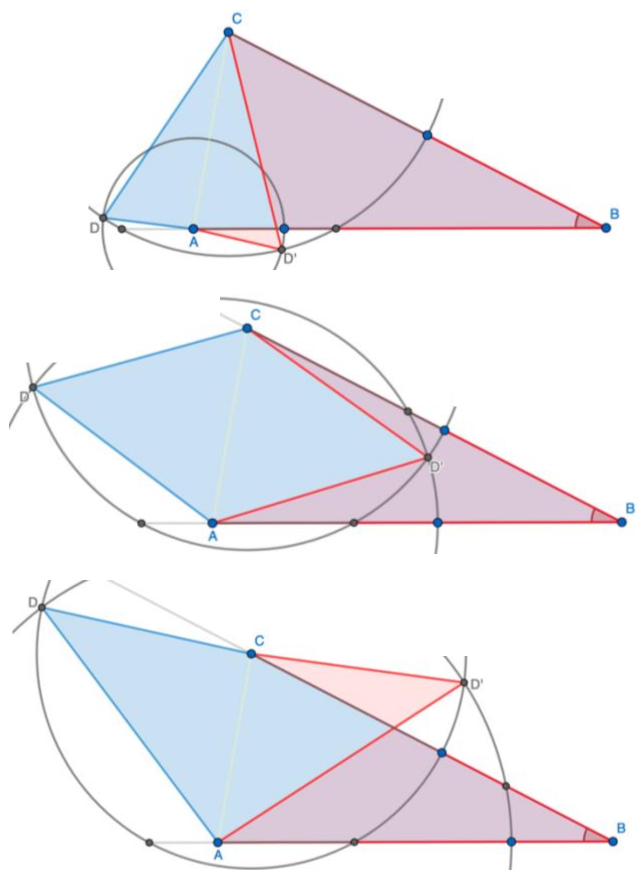
Or

$$\sqrt{4(S_1)^2 - 4(S_2)^2(\sin A)^2} - 2|S_4 - S_3 + S_2 \cos A| \geq 0$$

If *either* of them is true then the two quadrilaterals are congruent due to SSASS, and if there is a negative value under the root, then there is no intersection between the arc and the side to begin within this side, so the equation is false, but if the other equation is true, then the quadrilateral is unique and congruent to any quadrilateral with the same givens.

This result is amazing, let's test it on some cases! Here is a graph of how different S_1 values would differ the output of the two equations given the other inputs, if our equations are true, then both graphs should only be above the x-axis when the S_1 value results in a unique shape.





And it works! On the figures to the right are the same inputs while varying S_1, the shape is unique in two intervals, while $S_1 \leq 1.4$, the distance between A and the intersection to its right, and when $S_1 \geq 2.5$, the distance between point A and the intersection of S_4 (the radius of the circle centered at C) with S_3.

Developing a Pattern for Polygons in General

Triangles	Quadrilaterals	5-sided polygons	n-sided polygons
SAS	SASAS	SASASAS	S(AS× $(n-3)$)AS
ASA	ASASA	ASASASA	AS(AS× $(n-3)$)A
SAA	SASAA	SASASAA	S(AS× $(n-3)$)AA
SSA	SSASA	SSASASA	SS(AS× $(n-3)$)A
SSS	SSASS	SSASASS	SS(AS× $(n-3)$)S

The blue-shaded cells indicate the requirement of further conditions for congruency. As the number of sides increases these conditions get more and more complex, and that is an area of further research.

As it is clear from this table the pattern is developed using the basic building blocks of triangle congruency, in this paper we have developed and discovered wonderful and intricate patterns for polygons in general, with a fixed unique unit being highlighted in red, and attachments that complete the polygon uniquely written in black.

We can say that for a polygon with n-sides, it would require at least $2n-3$ congruent angles or sides to prove congruency, in the patterns in the table above.

Discussion and Conclusion

This paper's investigation illuminates the complex realm of polygon congruency and uniqueness, providing insights that go beyond the boundaries of mathematical theory. After looking at triangles and their well-known congruency standards, we delved into less-trodden areas like the Side-Side-Angle (SSA) scenario, which is sometimes ignored in the curricula We proved the validity of congruence theorems even in these less straightforward circumstances by rigorous geometric arguments, unveiling the fundamental ideas governing polygonal shapes.

And we have built upon that by expanding with tools we were using in triangles to help us in quadrilaterals. By dissecting the conditions under which quadrilaterals exhibit congruence, we uncovered patterns and relationships that enrich our understanding of geometric structures and we deduced and formulated the congruency theorems for them. And from these two types of polygons, we deduced a pattern for polygons in general.

The results of the SSASS and SSASA theorems and the general pattern for polygons were mentioned for the first time in this paper, I hope future research and analysis would result in formulating congruency theorems which are applied with restrictions beyond quadrilaterals.

In conclusion, this paper serves as a testament to the enduring relevance of geometry in our modern world. By unraveling the mysteries of polygon congruency and uniqueness, we not only deepen our appreciation for the elegance of mathematical reasoning but also empower ourselves to tackle the challenges of tomorrow with confidence and ingenuity.