1. Introduction

A prime number is often defined as a natural number that is greater than 1 that isn't a product of 2 or more smaller numbers. Now growing up in primary school, you might have been told that the distribution of prime numbers are random and thus an exact formula for prime does not exist. Well believe it or not, we're here to do one thing we've always desired – proving the teachers wrong. We are going to derive a formula for primes in this essay by walking through the journey C.P. willans did.

2. Baby steps

Picture yourself in 1964, writing for a mathematical gazette, trying to find an exact formula for prime numbers. Where do you start with? Luckily, Wilson's theorem has just came out. Stating that
n satisfies, $(n-1)!+1=0(\bmod n)$ if and only if n is prime or 1. In other words,
$\frac{(n-1)!+1}{n}$ would only be an integer if and only if n is prime or 1.

Now having this "prime detector" here, how could we engineer it into our formula for the Nth prime?

2.1 Thinking

Often the most optimal approach into solving a problem is by developing a "general Direction". Well here what is our general direction? Think about our prime detector. If we could detect whether or not a number is prime, we could in theory find the number of primes up to a certain number ***i*** ***.*** And with this, we have our new, "theoretical" function P(***i***), where it outputs the number of primes up to a certain number ***i*** , then we could construct a inequality of P(***i***)<***n*** , in other words, asking the question "is the number of primes up to ***i*** less than ***n*** ?" if we keep increasing our input ***i*** , eventually, the answer would be no. The first time this would be "no" would be when we've reached our Nth prime, and with that, we have our general direction. Great. But how do we get there?

2.2 Thinking outside the box….. and perhaps outside our setting of 1964

Now we have our "general strategy" of deriving the formula of the Nth prime, implementing it in the form of mathematical equations sounds and looks implausible. Perhaps we could cheat a bit and think in a moderner's perspective. The reason to why that this sounds so unconvincing and difficult is that right now, our "general direction" sounds a lot more like a flowchart with conditions and while loops than a sequence of arithmetic and trigonometric functions. In fact, ironically, this would have been a lot more pleasant if we are told to program this via coding than finding an mathematical formula. All of this could have been a GCSE coding assignment! Dire times are ahead of us, the idea of a “general formula” has never been so close yet so far from our reach, so visual yet so intangible. Or is it?

3. Improving our “prime detector”

Believe it or not, we could actually use our “flowchart” to help derive our formula. remember our Boolean “True” and “False”, this could be done as “1s” and “0s” in our prime detector function.

what we have:

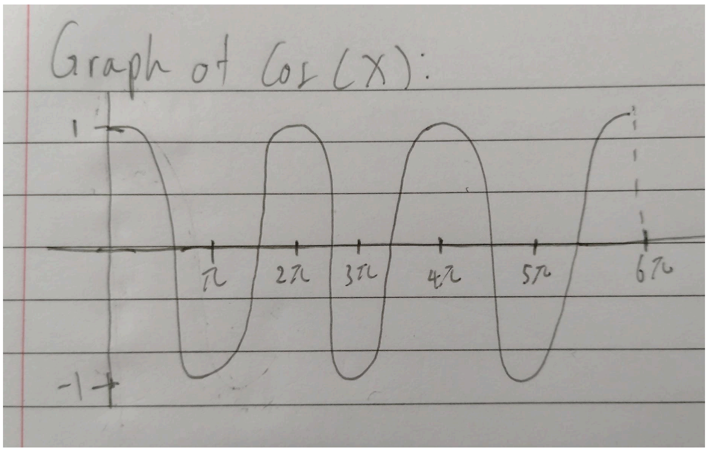
$$\frac{(j-1)!+1}{j} \text{ is } \begin{cases} \text{an integer} & \text{if } j \text{ is prime or } 1 \\ \text{some weird fraction} & \text{when } j \text{ is composite} \end{cases}$$

What we want:

$$\underline{\qquad\qquad} \text{ is } \begin{cases} 1 & \text{if } j \text{ is prime or } 1 \\ 0 & \text{if } j \text{ is composite} \end{cases}$$

The reality is often disappointing. What we wanted is quite far from what we have. Maybe we should start in small from making our desired function outputs 1 if j is prime.

What function gives 1 when we input an integer? The mighty cosine function comes to the rescue! Remember the cos graph? it is either 1 or -1 in $cos(n\boldsymbol{\pi})$



So now we have:

$$\cos\left(\frac{(j+1)!+1}{j}\,\pi\right) \text{ is } \begin{cases} \pm 1 & \text{when } j \text{ is prime or } 1 \\ \text{in the interval } (-1,1) & \text{when } j \text{ is composite} \end{cases}$$

Then by squaring our new function, we could effectively turn our -1 into 1, which makes life easier. So now we have:

$$\cos\left(\frac{(j+1)!+1}{j}\pi\right)^2 \text{ is } \begin{cases} 1 \text{ when } j \text{ is prime or } 1 \\ \text{in interval } [0,1) \text{ when } j \text{ is composite} \end{cases}$$

Now, we could then apply the floor function, which essentially makes everything between 0 and 1 zero. We have finally engineered our “improved” prime detector!

$$\left\lfloor \cos\left(\frac{(j-1)!+1}{j}\pi\right)^2 \right\rfloor \text{ is } \begin{cases} 1 \text{ if } j \text{ is prime or } 1 \\ 0 \text{ if } j \text{ is composite.} \end{cases}$$

3.1 making a giant step forward

Now with our improved prime detector, we can now proceed with our general direction and sum it over ***i***, (notice how our summation process now functions like a for loop, amazing!) which outputs the total number of primes smaller than or equal to ***i*** + 1 (note that our prime detector outputs 1 when j is 1) which is a version of our theoretical function P(***i***).

$$\sum_{j=1}^{i} \left\lfloor \cos\left(\frac{(j-1)!+1}{j}\pi\right)^2 \right\rfloor = (\text{the number of primes} \leq i) + 1$$

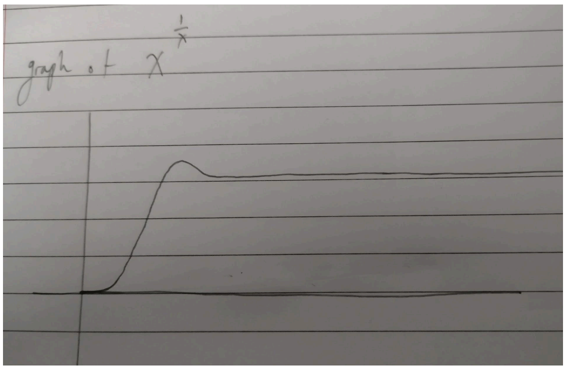
so this extremely complicated expression actually has a very straightforward meaning.

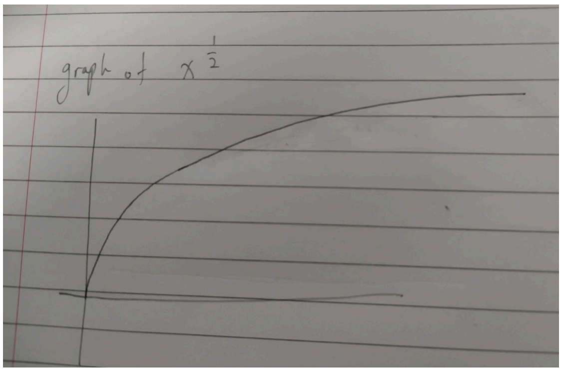
Having finally engineered this expression to our hands, we then need to construct the inequality based on our expression which does the following

$$\underline{\qquad\qquad} \text{ gives } \begin{cases} 1 \text{ if the nth prime} > i \\ 0 \text{ if the nth prime} \leq i \end{cases}$$

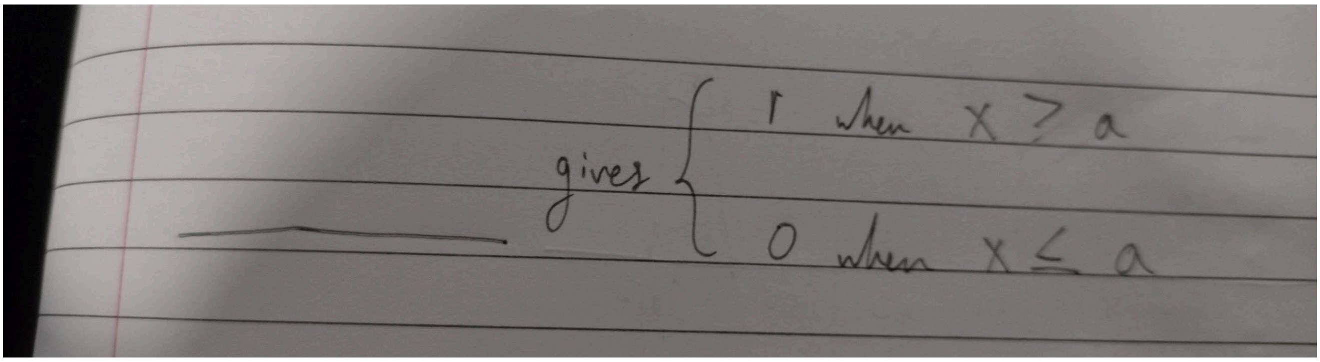
Here we could think of a number ***a*** as a certain threshold. since we're again dealing with the boolean like 1s and 0s, it's to experience that we might need to take the floor function again.

then, we have to think – what function is initially increasing, but then plateaus after a certain point? if you guessed something like $x^{\frac{1}{2}}$ or $x^{\frac{1}{3}}$ you would be very close. The problem of these is that even though their graph looks like it plateaus, they are still divergent. It is only $x^{\frac{1}{x}}$ that converges quick enough.

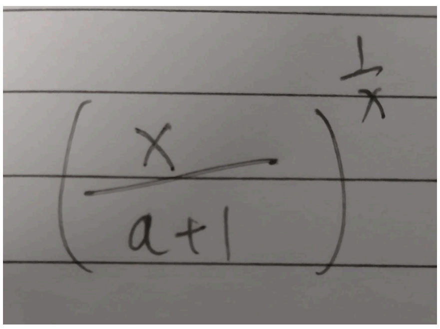




So now we have our base function being $x^{\frac{1}{x}}$, we would need to modify this such that:



since we are only considering integer values of x and a and this expression should only be equal to 1 when x>a (i.e the expression should only be >= 1 at minimum x=a+1) we should be able to conclude that our expression should look like:



then, by finally taking the floor function, we would obtain:

$$\left\lfloor \left(\frac{x}{a+1}\right)^{\frac{1}{x}} \right\rfloor \text{ gives } \begin{cases} 1 & \text{when } x > a \\ 0 & \text{when } x \leq a \end{cases}$$

afterwards, we could rewrite a in the threshold we wanted, i.e. the no. of primes up to ***i*** and x into n since we're gonna do a summation over it. This gives us :

$$\left\lfloor \left(\frac{n}{(\text{no. of primes} \leq i) + 1}\right)^{\frac{1}{n}} \right\rfloor \text{ gives } \begin{cases} 1 & \text{when } n > (\text{no. of primes} \leq i) \\ 0 & \text{when } n \leq (\text{no. of primes} \leq i) \end{cases}$$

in which we could replace our function into the once we've found in part 3.1, and also, the condition (when n > (no. of primes <=i)) is equivalent to (when the nth prime is greater than i) this form is preferred as we are here varying n and with a fixed i. so after a it of modification, we get:

$$\left\lfloor \left(\frac{n}{\sum_{j=1}^{i} \left\lfloor \cos^2\left(\pi \frac{(j-1)!+1}{j}\right) \right\rfloor + 1}\right)^{\frac{1}{n}} \right\rfloor \text{ gives } \begin{cases} 1 & \text{when } i < \text{nth prime.} \\ 0 & \text{when } i \geq \text{nth prime.} \end{cases}$$

which is indeed the expression we've been aiming for.

3.2 finding a bound for our nth prime

Now, we can take a sign of relief as we have gone through the hardest part of our equation. Now, all we have to do is to do a summation over it in which the upper limit has to be above the nth prime we're looking for such that we don't end up not adding enough 1s before ending the summation. Here's where Bertrand's postulate comes in handy.
the Bertrand's postulate states that:

for every n > 3, there is a prime p satisfying n < p < 2n

in which here we could just sum it over 2^n , and then, taking account to the fact that our sum gives zero as we reach the nth prime, we need to compensate for this by adding 1 to our entire formula. giving us this beast of a formula

$$p_n = 1 + \sum_{i=1}^{2^n} \left\lfloor \left(\frac{n}{\sum_{j=1}^{i} \left\lfloor \left(\cos \frac{(j-1)!+1}{j} \pi \right)^2 \right\rfloor} \right)^{1/n} \right\rfloor$$

and viola! We have completed our journey for finding a mathematical expression for the nth prime. Now drink a cup of tea and take a rest as we've done the unthinkable

4. Conclusion

Well that was one hell of a journey. We've went from using griefing over our seeming useless flowchart into implementing it into our mathematical formula via some crazy arithmetic and trigonometric functions. Now could our hard engineered formula potentially change the worlds view on primes and potentially win the Shaw's prize for maths? Sadly no. No matter how fascinating this formula might seem, it has little to no practical use. C.P. willans formula not only involves a nested sum, but a factorial inside to sum which makes it take extremely long to compute primes. In fact, computing the 50th prime would consist of 1+1+1+.......+1 229 times and computing to add millions of zeros afterwards. You can imagine how much it goes up exponentially as the numbers grow.

5. Is this the end?

This is a bit of a food for thought after all this journey we've went through. Whilst this is a formula, this could compute primes, this is so long and hard to compute that it holds no practical value. As Herbert wilf said in his article "what is an answer" , ***sometimes we find the solution that is so messy and long, full of sign alterations and whatnot, that we may feel that the disease is preferable to the cure***. This formula certainly isn't an answer to the question of what's the nth prime in his perspective. One might wonder after all this hard work, all we get is one useless formula. What's the point of this? Well in my perspective, this formula itself isn't the point, it's how we get there. It's about the depth of how much simple arithmetic functions can express and functions. You would never expect how a summation function could work like a "for loop" don't you? Sometimes the point of a journey isn't about the destination, but the astonishing views we've seen along the way.