The Fourier Series: A Brief Introduction to Fourier Analysis

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1 Introduction

In 1822 Jean- Baptiste Joseph Fourier, a French mathematician and physicist, published his famous work, *The Analytical Theory of Heat.* He explored the concept and mathematical representation of heat transfer between two bodies, based off of some of Newton's previous work. Unbeknownst to him at the time, he also pioneered one of the most influential spheres of mathematics of the modern world, extremely relevant to areas like Engineering, Physics or even Computer Science.

Joseph Fourier was the father of ***Fourier Analysis***, a topic in which I hope to provide a brief yet intriguing glimpse into, from a mathematics perspective.

The core idea behind Fourier Analysis, is the idea of a ***Fourier Series***, the representation of a function as an infinite sum of some variation of trigonometric functions; that is to say, sines and cosines. This may seem quite random,and at first glance, its applications are non-trivial, but over hundreds of years it has proven its usefulness time and time again.

For one, its original purpose. Fourier Analysis greatly simplifies the study of heat transfer, exactly what Fourier set out to do in his manuscript.But in a more modern setting, the idea of the Fourier series is still greatly useful. The modern world runs on waves, sound waves for talking with each other, radio waves for your telephones, infrared waves for your TV remotes, waves are everywhere. You should recall from early studies of trigonometry in school, that sine and cosine are mathematical representations of waves, functions that oscillate up and down between a maximum and a minimum.A fairly simple idea. Since waves are so prevalent to our day to day lives, especially in technology, we need a way to manipulate them to our advantage.Let's look at a brief example.

Imagine you are recording a video next to something very noisy, a main road perhaps. You probably don't want to hear the loud droning noises of cars rushing by as you're trying to record your voice, for example. But a lot of modern audio editing tools can clean this noise up for you will little effort. How do they do that? It turns out, by decomposing the raw sound waves into lots and lots of sine and cosine waves, all with different frequencies and amplitudes (a process called ***Fourier Transformation***, we can simply remove the waves that have frequencies and amplitudes that we don't like, the noises of the cars in this case, and then rebuild the new, desired sound (***Fourier Synthesis***). This is just one of many applications of Fourier Analysis.

In this essay, I hope to break down the derivation of the Fourier Series, and explain how to turn a function into a sum of trigonometric functions. It may sound daunting, but not to worry, it's much easier than it sounds.

2 The Theory

In the Introduction, I touched on the Key idea of a Fourier Series, but formally, it states that: ***Any periodic function can be expressed as an infinite series of trigonometric functions***. There are a few things that are worth discussing more. Firstly, what is a periodic function? Given the name, if you guessed that it probably has something to do with the function repeating itself ***periodically***, and you would be correct. One of the simplest and most well known periodic functions is the **sine function**

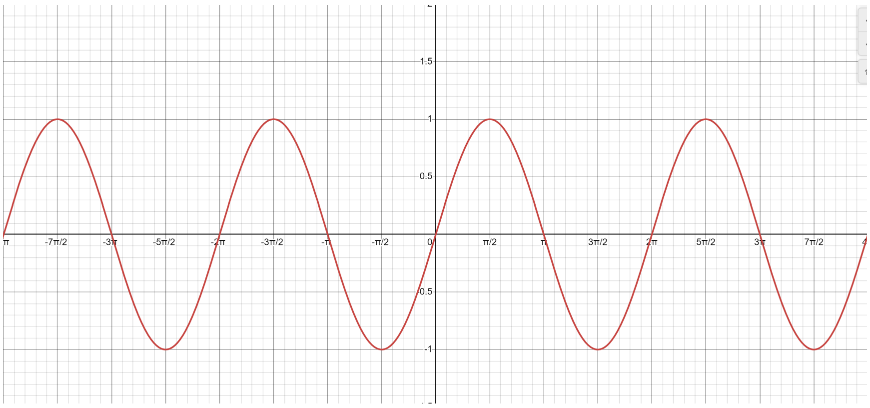

Figure 1: sin(x)

We can see that this function repeats itself, oscillating between 1 and -1 every 360 degrees or 2π radians. We can use this idea to formally define a periodic function. ***A periodic function is a function that repeats itself on an interval with width*** L The sine function repeats itself every 360 degrees, so in the case of sine, $L = 2\pi$ (2π radians is 360 degrees).

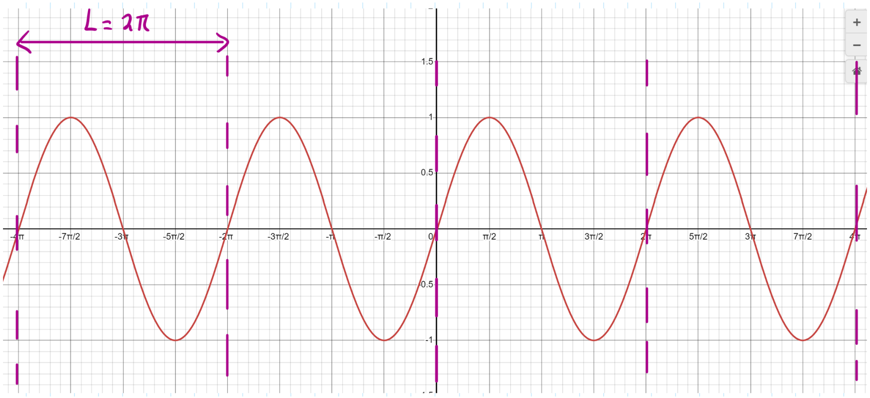

Figure 2: Sin(x) has interval $L = 2\pi$

It is clear to see that every area between each set of purple lines is exactly the same as each other, thus showing the periodic nature of the function. It is also worth noting that this is true for any section of the function with width 2π. This is all well and good, but as we know, ***sine is already a trigonometric function***; but what about other periodic functions, that aren't trigonometric?

Consider this ***square wave***. This isn't even close to a trigonometric function, but we can clearly see

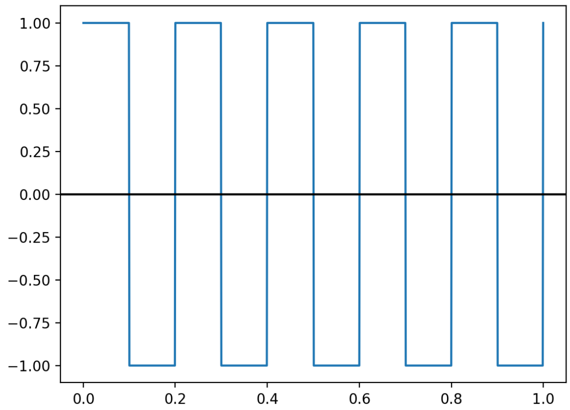

Figure 3: A square Wave

that it's periodic. It jumps between 1 and -1 forever. In fact, the square wave cannot be explicitly defined as a mathematical function, but we can ***approximate it***. This process is usually easier to understand by reverse engineering it. Below is the formula for a Fourier Series.

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos(\frac{2\pi nx}{L}) + b_n \sin(\frac{2\pi nx}{L})) \tag{1}$$

But what is this monstrous looking formula? Let's start by making sure we understand each piece of it.

2.1 Summations

what does this big Greek letter $\sum_{n=1}^{\infty}$ mean? What are the numbers on the top and bottom? It is quite simple really. n starts from 0 and is incremented by 1 until it reaches the number on the top. Every time we do this, we add up the expression inside with the relevant value of n. Consider this:

$$\sum_{n=1}^{n=5} n = 1 + 2 + 3 + 4 + 5 = 15$$

As you can see, we add up n but for every term we add, the value of n increases by 1. Similarly:

$$\sum_{n=1}^{n=3} 2^n = 2^1 + 2^2 + 2^3 = 12$$

and one more example:

$$\sum_{n=1}^{n=4} \sin(n\pi) = \sin(\pi) + \sin(2\pi) + \sin(3\pi) + \sin(4\pi) = 0$$

2.2 Trigonometry

Lets look at what the trigonometric terms represent. Consider:

$$sin(x) \tag{2}$$

As we already know, $sin(x)$ has period of $L = 2\pi$, but what if we want to **specify the period?** To make life easy for ourselves, lets **compress** the function so it has period 1. We can do this simply like by multiplying x by 2π:

$$sin(2\pi x) \tag{3}$$

What have we just done here? Usually, once $x = 2\pi$, $sin(x) = 0$. By multiplying x by 2π, x now only needs to be 1 to obtain $sin(2\pi) = 0$. We have therefore **compressed** the function so it has period 1. We can see the use of this in the next step:

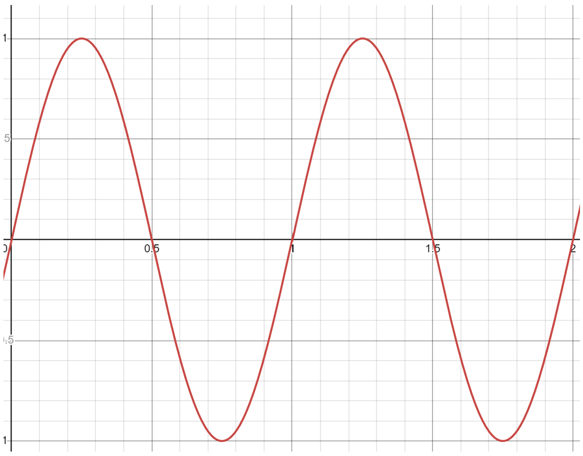

Figure 4: A sine graph with period 1

$$sin(\frac{2\pi x}{L}) \tag{4}$$

We have done this so that once $x = L$, we once again obtain $sin(2\pi) = 0$. In a very similar chain of thought to the last step, we have now defined the period of our function to be $= L$, where we are free to choose a value of L. And finally, we can specify the **frequency**; the number of oscillations every given period, by multiplying by n, to give us:

$$sin(\frac{2\pi nx}{L}) \tag{5}$$

Where n is a whole number, representing the number of oscillations.

This same chain of steps holds for the cosine term as well. If we now bring this back into context,If we add up terms as n is increased from 1 all the way to infinity, we see that we are adding sine and cosine terms with a **fixed period** and **increasing frequency**; since the frequency is controlled by n. We haven't got the full story just yet, we still don't know how to calculate the initial term a_0 and the **coefficients**; a_n and b_n, but at this point, it's worth taking a look at what this formula actually does. Lets go back to our square wave, For now, you'll just have to trust me on the fact that the Fourier series for the specific square wave that oscillates between 1 and -1 is:

$$\frac{4}{\pi}sin(\pi x) + \frac{4}{3\pi}sin(3\pi x) + \frac{4}{5\pi}sin(5\pi x) + \frac{4}{7\pi}sin(7\pi x)... \tag{6}$$

This is an infinite series. Lets look at what happens as the number of terms increases.

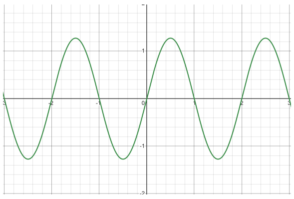

Figure 5: 1 term

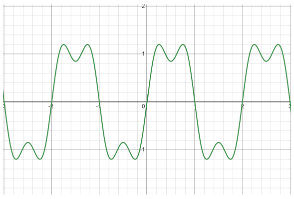

Figure 6: 2 terms

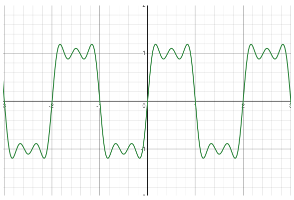

Figure 7: 3 terms

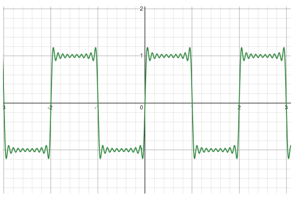

Figure 8: 10 terms

And would you look at that! Its clearly shown that as we increase the number of terms our approximation gets better and better, slowly approaching a "pure" square wave. This is quite nice

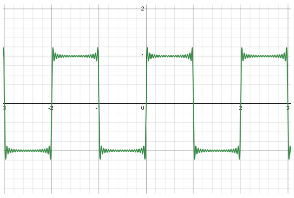

Figure 9: 20 terms

for us, as we have explicitly shown the original conjecture, we have **written a periodic function as the sum of trigonometric functions** We can't pat ourselves on the back for too long, however, as we are yet to find the ***coefficients***. Earlier, I explicitly stated the formula for the series of a square wave,but how did we actually arrive there? Why are there no cosine terms? To answer this, we're going to need to draw on some more theory,namely a few important integrals.

2.3 Some Useful Integrals

You may think this is quite a random aside, but I assure you it will be very relevant to what we're trying to achieve. Before I start, even if you don't have much knowledge of what Integration is or how it works, there's really only one important fact to remember. ***The value of an integral of a function between two points, is equal to the area under the curve between the same two points*** If the line is below the x-axis, the result of the integral will be **negative**. This is important, so remember this.

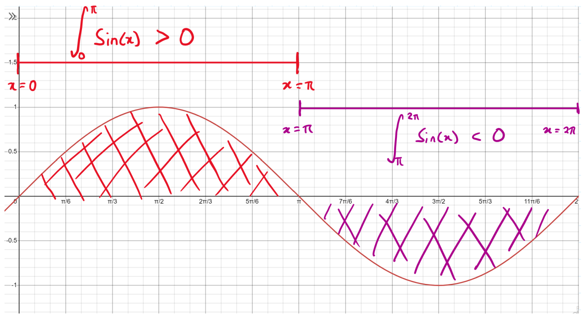

Figure 10: The Integral is the Area under a curve

$$\int_0^L \sin(\frac{2\pi nx}{L})\cos(\frac{2\pi mx}{L})\,dx$$

Lets get started.Consider the integral above.We see that the expression inside the integral, the *integrand* is the product of two very familiar functions. Just notice that the sine term contains an n, while the cosine term contains and m, where n and m can be any positive integers. This shows that the functions do not have the same frequency, except in the case of $n = m$. Also that the integral is taken over 0 to L, the period. in other words, we are finding the area under the curve through ***a whole number complete oscillations***. This integral may look complicated to actually evaluate, but we can actually evaluate it purely with some good mathematical reasoning. Lets look at the graph of $\sin(x)\cos(x)$ (next page) specifically between 0 and L, where $L = \pi$ in this example:

Clearly, the graph has nicely done a single,full oscillation. So what happens if we integrate this over

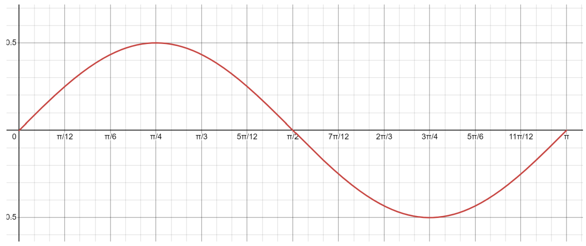

Figure 11: sin(x)cos(x) between 0 and L

its period? Well, if we remember that the integral just gives us the area between the curve and the x-axis (where the result is negative if the curve is below the x-axis), we can see that because the curve has gone a complete oscillation, *the area above and below the x axis is exactly the same.* Since above the x axis is positive, and below the x axis is negative, if we add these together we get 0.This is quite nice for us. OK, but what about n and m? How do they affect this idea? They are distinct two different numbers after all.To answer this, we're going to make use of some **trigonometric identities**. You don't need to know where they come from or how they're derived, just know that these relationships are always true. In this case we will use the **product to sum identities**, that state:

$$\sin(a)\cos(b) = \frac{1}{2}(\sin(a+b) + \sin(a-b)) \tag{7}$$

$$\sin(a)\sin(b) = \frac{1}{2}(\cos(a-b) - \cos(a+b)) \tag{8}$$

$$\cos(a)\cos(b) = \frac{1}{2}(\cos(a+b) + \cos(a-b)) \tag{9}$$

Lets apply this to the integrand, with $a = \frac{2\pi nx}{L}$ and $b = \frac{2\pi mx}{L}$. We get:

$$\int_0^L \frac{1}{2}(\sin(\frac{2\pi nx}{L} + \frac{2\pi mx}{L}) + \sin(\frac{2\pi nx}{L} - \frac{2\pi mx}{L}))\,dx$$

And if we simply factor out $\frac{2\pi x}{L}$, from the insides of both trigonometric terms, we get:

$$\int_0^L \frac{1}{2}(\sin((n+m)\frac{2\pi x}{L}) + \sin((n-m)\frac{2\pi x}{L}))\,dx$$

Both terms in the integrand still complete a whole number of oscillations, (since n and m are both positive integers), but we just need to be wary of the special case $n = m$. Luckily for us, the integral still turns out to be zero as the second sine term becomes $\sin(0) = 0$ when $n = m$. Lets do this process for two more integrals:

$$\int_0^L \cos(\frac{2\pi nx}{L})\cos(\frac{2\pi mx}{L})\,dx$$

This one contains two cosine terms. By using our product to sum identities and factorising in the same way as before, we get:

$$\int_0^L \frac{1}{2}(\cos((n+m)\frac{2\pi x}{L}) + \cos((n-m)\frac{2\pi x}{L}))\,dx$$

We have to check the special case again. This time when $n = m$ the second term becomes $\cos(0) = 1$ (the first term still becomes 0 as it completes a whole number of oscillations). If this happens, the integral will essentially become:

$$\frac{1}{2}\int_0^L 1\,dx$$

We've just taken the half outside the integral there, but his is quite literally the easiest integral we could ask for.It is evaluated as such:

$$= \frac{1}{2}[x]_0^L = \frac{L}{2}$$

So the above integral is imply equal to 0 when $n \neq m$ and $\frac{L}{2}$ when $n = m$. We have one final integral:

$$\int_0^L \sin(\frac{2\pi nx}{L})\sin(\frac{2\pi mx}{L})\,dx$$

This one with two sine terms. Again, by following the same process, we obtain:

$$\int_0^L \frac{1}{2}(\cos((n-m)\frac{2\pi x}{L}) - \cos((n+m)\frac{2\pi x}{L}))\,dx$$

And similarly by checking $n = m$, the first term becomes 1 and the other is always 0. In exactly the same was as the last one:

$$= \frac{1}{2}[x]_0^L = \frac{L}{2}$$

There is a special name for functions that behave this way. Two functions $f(x)$ and $g(x)$ are said to be ***Orthogonal*** if:

$$\int_a^b f(x)g(x) = 0$$

2.4 The Coefficients

Alright, now that we've established this, how does it help us? If you remember, we were on a quest to try and find the **missing coefficients** of the Fourier Series, but just spent two pages working with seemingly unrelated integrals. Let's begin to tie all this theory together. Let's recall our original equation, way back from the start.

$$f(x) = a_0 + \sum(a_n\cos(\frac{2\pi nx}{L}) + b_n\sin(\frac{2\pi nx}{L}))$$

Lets begin by tackling a_0. As it turns out, all the coefficients including a_0 will be **defined in terms of** $f(x)$; in other words, the function itself will affect what the coefficients of its Fourier series are. Let me show you what I mean. The integrals we just showed were not, in fact, unrelated. Look at what happens if we **integrate both sides across** 0 **and** L

$$\int_0^L f(x) = \int_0^L (a_0 + \sum(a_n\cos(\frac{2\pi nx}{L}) + b_n\sin(\frac{2\pi nx}{L})))$$

Can you start to see what we're getting at here? What happens to the trigonometric terms as we integrate over their period. You guessed it, **they all become** 0. Thus, we are left with:

$$\int_0^L f(x) = \int_0^L a_0$$

Since a_0 is a constant, we can easily evaluate this:

$$\int_0^L f(x) = [a_0]_0^L = a_0 L$$

And therefore:

$$a_0 = \frac{1}{L}\int_0^L f(x)$$

To find the a_n and b_n coefficients, we again use the theory we established earlier, in quite a clever way. What happens if we multiply both sides by $\cos(\frac{2\pi mx}{L})$? Well, every single term in the summation will be multiplied by it, and thus will all integrate to zero, as we've shown before. EXCEPT for when $n = m$. Remember how a summation works? Each term is created by plugging in an integer value of n, adding onto the sum and then incrementing n.Therefore, if we are summing from 0 to ∞ and m is some positive integer, there will *always* be the case that one term will satisfy the case $n = m$. Therefore:

$$\int_0^L f(x)\cos(\frac{2\pi mx}{L}) = \int_0^L a_n \cos(\frac{2\pi nx}{L})\cos(\frac{2\pi mx}{L})$$

Since a_n is a constant, we can factor it out of the integral and evaluate it. If we recall that $\int_0^L \cos(\frac{2\pi nx}{L})\cos(\frac{2\pi mx}{L})\,dx = \frac{L}{2}$ for $n = m$, we get

$$\int_0^L f(x)\cos(\frac{2\pi mx}{L}) = a_n\frac{L}{2}$$

and thus:

$$a_n = \frac{2}{L}\int_0^L f(x)\cos(\frac{2\pi nx}{L})$$

And using the exact same logic to find b_n, except we multiply by $\sin(\frac{2\pi mx}{L})$, we get:

$$b_n = \frac{2}{L}\int_0^L f(x)\sin(\frac{2\pi nx}{L})$$

Note: I have replaced m with n in both cases, to avoid confusion. It will intuitively make more sense to use n, since $n = m$ anyway. Now we have all the pieces of the puzzle, it's time to see how it all works together. Let's compute the Fourier series of a square wave that oscillates between 0 and 1 periodically, right from start to finish. First of all, how do we represent a square wave as a function? Lets define our square wave to be:

$$f(x) = \begin{cases} 1, 0 < x < \pi \\ 0, \pi < x < 2\pi \end{cases} \qquad f(x + 2\pi) = f(x) \tag{10}$$

We have made a ***piece-wise function***, that is, a function defined on a sequence of intervals.All we've said here is that $f(x)$ will equal 1 when x is between 0 and π, and equal 0 when x is between π and 2π. We have also made $f(x)$ periodic with $L = 2\pi$, thus representing our square wave. Now let's use this to calculate a_0, a_n and b_n, with $L = 2\pi$ and $f(x)$ being defined above.

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)$$

We can take advantage of the nature of $f(x)$ here. We know that between π and 2π, $f(x) = 0$. Therefore, wouldn't it be nicer if we split the integral into two parts and see what happens?

$$a_0 = \frac{1}{2\pi}(\int_0^{\pi} f(x) + \int_{\pi}^{2\pi} f(x)) = \frac{1}{2\pi}(\int_0^{\pi} f(x) + \int_{\pi}^{2\pi} 0) = \frac{1}{2\pi}\int_0^{\pi} f(x)$$

We have effectively cut the integral in half, and now we only need to consider $f(x)$ for x between 0 and π, which we have already defined to be 1. Therefore we have:

$$a_0 = \frac{1}{2\pi}\int_0^{\pi} 1 = \frac{1}{2}$$

Nicely done. Now similarly for a_n and b_n we can use the trick of only considering the first half of the limits:

$$a_n = \frac{2}{2\pi}\int_0^{\pi} 1\cos(\frac{2\pi nx}{2\pi})$$

And with some simplifying:

$$a_n = \frac{1}{\pi}\int_0^{\pi} \cos(nx) = [\frac{1}{\pi}\frac{1}{n}\sin(nx)]_0^{\pi} = \frac{1}{\pi}\frac{1}{n}\sin(n\pi)$$

Since n is a whole number, the sine term will be 0 for any value of n. This is because sine of a whole number multiple of π is always 0. Therefore $a_n = 0$ for all values of n. Now for the b_n term.

$$b_n = \frac{1}{\pi}\int_0^{\pi} \sin(nx) = [-\frac{1}{\pi}\frac{1}{n}\cos(nx)]_0^{\pi} = -\frac{1}{n\pi}(\cos(n\pi) - 1)$$

This integral is a little bit more interesting. When cosine takes an **even multiple of** π it is equal to 1, but when it takes an **odd multiple of** π it is equal to -1. Therefore if n is even, $\cos(n\pi) = 1$ and $b_n = 0$. But if n is odd $\cos(n\pi) = -1$ and $b_n = \frac{2}{n\pi}$. What does this look like when we expand our original definition for a Fourier Series?

$$f(x) = a_0 + \sum(a_n\cos(\frac{2\pi nx}{L}) + b_n\sin(\frac{2\pi nx}{L}))$$

For each term, we plug in the value of n and hence a_n and b_n. n starts from 1 and is continually incremented to infinity. Lets see this in action.

$$f(x) = \frac{1}{2} + 0 + \frac{2}{\pi}\sin(x) + 0 + \frac{2}{3\pi}\sin(3x) + 0 + \frac{2}{5\pi}\sin(5x)....$$

Here we can see the series in full. Notice how there are no cosine terms, because a_n is always 0, as we've shown earlier, and that any term where n is even becomes 0. Because there is clear pattern here, we can nicely compact this equation to look like this:

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n-1)}\sin((2n-1)x)$$

Let's visualise this in the same way we did before.

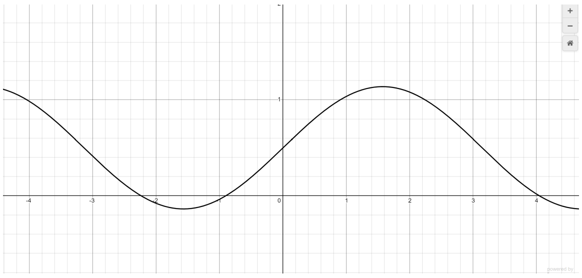

Figure 12: 1 term

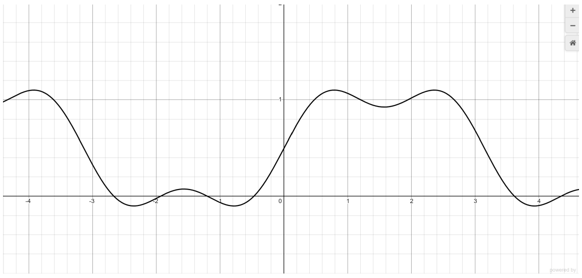

Figure 13: 2 terms

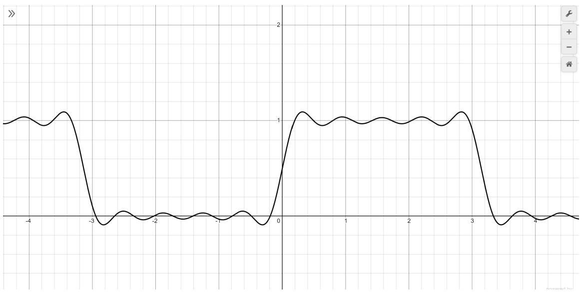

Figure 14: 5 terms

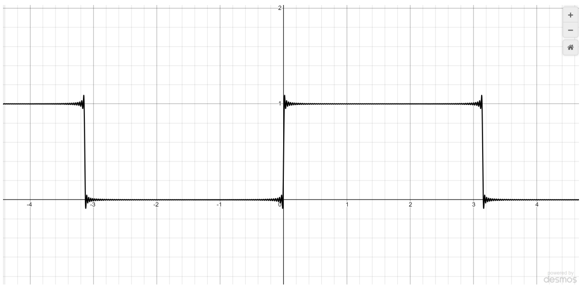

Figure 15: 40 terms

2.5 Conclusion

And thus we've computed the Fourier series for a square wave from start to finish. To recap, we explored how to decompose any periodic function into an infinite series of trigonometric functions, the foundation of Fourier Analysis. While this essay gave an introduction to the topic, the journey doesn't need to end here. Fourier Analysis is an incredibly expansive topic, and its study will open up doors into many exciting areas of Science, technology, engineering and mathematics. I hoped I've been able to spark a new passion for any aspiring mathematicians and beyond. Thank you for reading.