How Powerful Is The Death Star?

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On first hearing the fantastical term 'dimensional analysis,' it is natural to immediately cast your mind towards sci-fi films: warp drives, flux capacitors, portal guns and the like. Leading to immense disappointment upon the realisation that the topic seems to be a fancy rebranding of the classic quote given by every primary school teacher: 'Don't forget your units!', 'How many whats? Bananas?'. However, I'd like to investigate beneath the surface level of this topic, and reveal some of the interesting applications of this technique.

1 What is Dimensional Analysis?

As previously mentioned, we must use units for measurements. It wouldn't make sense to measure the height of an antelope in terms of apples, however, it would make more sense to talk about this measurement in centimetres, or feet, or even plank lengths. These units all share the dimension of length or L. The notation used for this is a pair of square brackets.
I.e., [Height of an antelope] = L - a length.

There are seven primary dimensions from which other measurements are comprised shown below.

Quantity	Length	Mass	Time	Temp	Current	Light	Matter
Dimension	L	M	T	K	I	C	N
Base unit	Metre	Kilogram	Second	Kelvin	Ampere	Candela	Mole

Table 1: Base units and their dimensions.

These units are the building blocks from which all other units are made. For instance, speed (v) is measured in metres per second.

$$v = \frac{s}{t} \implies [v] = LT^{-1} \tag{1}$$

1.1 Dimensional Homogeneity

Homogeneity is yet another exciting word which sounds like it could hint towards teleportation or time travel. However, the reality is that the word comes from the Greek homogenous meaning ‘same type’. Dimensional homogeneity effectively means that both sides of an equation describing a physical phenomenon must be coherent. That is, the units of the left side must be the same as those of the right. We can see this principle in action by considering any of the equations we use to describe the mechanics of a physical system. Take, for example, this beloved ‘suvat’ equation.

$$s = ut + \frac{1}{2}at^2 \tag{2}$$

This particular equation relates displacement to velocity, time and acceleration. The following is a dimensional analysis of the equation:

$$[ut] = [u][t] = LT^{-1}T = L \qquad [\frac{1}{2}at^2] = [a][t]^2 = LT^{-2}T^2 = L \tag{3}$$

$$[s] = [ut] + [\frac{1}{2}at^2] = L \tag{4}$$

We can disregard constants as they are dimensionless and therefore we are left with a homogenous statement; as we hoped displacement appears to have the dimensions of a length.

1.2 Rayleigh's method

Lord Rayleigh[1] developed a method for finding a relationship between variables by using dimensional analysis. Given a dependent variable D, we guess the independent variables that will influence the result. Let's call them $I_1, I_2, \cdots, I_n$. Rayleigh argued that D must be a function of these Is, and so, he developed an equation of the following form.

$$D = CI_1^a I_2^b \cdots I_n^m \tag{5}$$

Where C is a dimensionless constant and a, b, ..., m are arbitary exponents. We can now express the variables in terms of their primary dimensions, and create simultaneous equations to solve for the exponents.

[1] Not only does he have a cool name, he also proved that the sky is blue using this method!

1.3 A Simple Example:

In order to show Rayleigh's method in action, I decided to try to create a situation in which the technique could be tested. Below is my doodle of a satellite orbiting a planet.

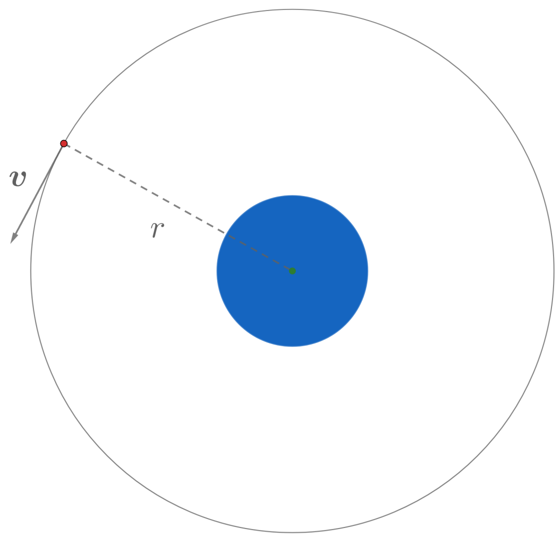

Figure 1: A sketch of a satellite orbiting a planet.

I would like to find a relationship to describe the orbital velocity v of this satelite, and so we will select this as the dependent variable. It is reasonable to assume that the main independent variables that will influence this velocity are the radius of the orbit r and the acceleration due to gravity g. Therefore, I will suggest that v is a function of r and g.

$$v = Cr^{\alpha}g^{\beta} \tag{6}$$

$$[v] = [Cr^{\alpha}g^{\beta}] = L^{\alpha}(LT^{-2})^{\beta} \qquad LT^{-1} = L^{\alpha+\beta}T^{-2\beta} \tag{7}$$

From (7) we can now create a system of simultaneous equations for α and β.

$$\begin{cases} \alpha + \beta = 1 \\ 2\beta = 1 \end{cases} \tag{8}$$

By (8) we see $\alpha = \beta = \frac{1}{2}$. Therefore, it has been shown that:

$$v = C\sqrt{rg}$$

To check the validity of this relationship, it can be observed that this is the two body problem. The velocity is described by Kepler's third law.

$$v = \sqrt{\frac{GM}{r}}$$

Where G is Newton's constant of gravitation and M is the mass of the planet. We know that $g = \frac{GM}{r^2}$ therefore substituting into my equation we find that $v = C\sqrt{\frac{GM}{r}}$, it appears the relation is correct and the constant C is actually equal to one. Kepler only beat me to this discovery by 406 years, and admittedly he did have a headstart.

2 Taylor and the Trinity Test

If you've not been living under a rock for the past few months, you would likely be aware of Cillian Murphy's Oscar-winning performance as Oppenheimer in the film of the same name. The events of the film follow the development of the first nuclear bomb including the first Trinity Test which took place on the 16th of July 1945 in New Mexico.

Figure 2: Oppenheimer and co. observing the remnants of the test site.

Of course, the information surrounding this test and the Manhattan Project in general was highly classified. Therefore you can imagine that the CIA might have had a few questions when G.I. Taylor accquired a remarkably accurate estimate for the blast's energy. His method? Dimensional analysis and some photos from Life magazine[2].

2.1 Taylor's Method

Taylor's first step was his biggest, as is often the case, the hardest part of this puzzle was finding the right pieces. From all of the possible independent variables, Taylor assumed that the bombs energy (E), the density of the air (ρ) and the time elapsed since the explosion (t) would combine to give the radius (r) of the explosion.

$$r = CE^{\alpha}\rho^{\beta}t^{\gamma} \tag{9}$$

Taylor then proceeded to conduct a dimensional analysis as follows.

$$[r] = [CE^{\alpha}\rho^{\beta}t^{\gamma}] = (ML^2T^{-2})^{\alpha}(ML^{-3})^{\beta}(T)^{\gamma} \tag{10}$$

$$L = M^{\alpha+\beta}L^{2\alpha-3\beta}T^{\gamma-2\alpha} \tag{11}$$

He could now solve a system of simultaneous equations.

$$\begin{cases} \alpha + \beta = 0 \\ 2\alpha - 3\beta = 1 \\ \gamma - 2\alpha = 0 \end{cases} \tag{12}$$

Resulting in: $\alpha = 1/5, \beta = -1/5, \gamma = 2/5$. If we substitute these values into the initial equation (9) we see that:

$$r = CE^{1/5}\rho^{-1/5}t^{2/5} \tag{13}$$

We can rearange this equation for E as follows.

$$E = C'\frac{r^5\rho}{t^2}$$

[2]This may not be entirely true, Taylor was actually present at these tests, but it makes for a fun story.

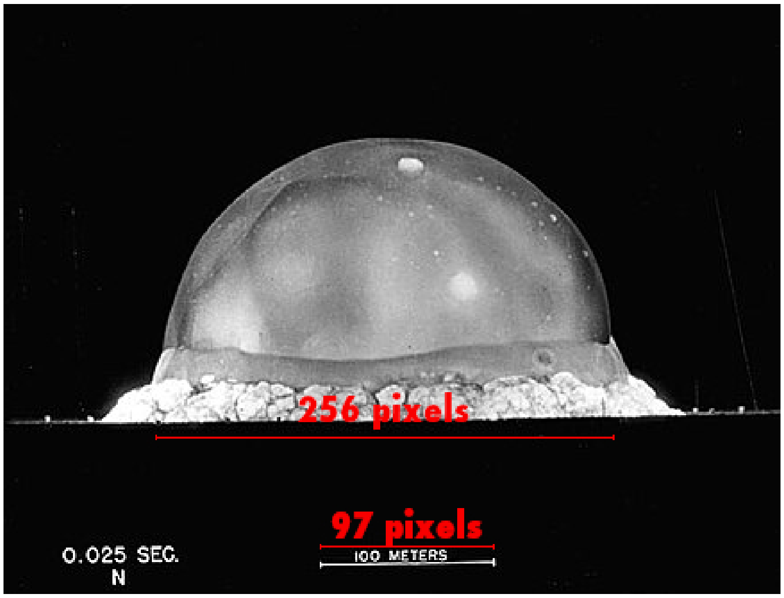

Figure 3: Trinity Test fireball with pixel measurements.

Where C' is the result of the algebraic manipulation upon constant C. Taylor had experimental evidence to show that for other explosions C' is very close to 1 in air. We now have an equation for E in terms of known variables. ρ is around 1.225kg/m^3, and the time and radius of explosion are dependent on the image used for estimation. For example, let's choose an image of the explosion at 0.025s. I have calculated the width in pixels of the explosion's diametre and the handy conversion bar in figure 3.

From this image we can calculate the radius as $r = \frac{100}{97} \times 256 \times \frac{1}{2}$ or $r = 132$m. Substituting these values into our equation for E we can show that.

$$E = \frac{132^5 \times 1.225}{0.025^2} = 8 \times 10^{13} J \tag{14}$$

Reports have placed the actual yield of the bomb as around 21 kilotons of TNT (which is somewhat ridiculously the standard unit for these measurements). One kiloton of TNT gives off around 4×10^{12}J of energy, so 21 Kt would translate to about 8.4×10^{13}. When compared to our calculations. That result gives a percentage error of $\frac{8.4\times10^{13} - 8\times10^{13}}{8.4\times10^{13}} \times 100$. That's 4.8%! Remarkably accurate for some pretty simple calculations.

3 A leap into space

Taylor's findings (along with the self-similar discoveries of Sedov and Von Neumman) reveal a power law shown in (13). The radius of the explosion is modeled as a constant $(\frac{E}{\rho}^{1/5})$ times $t^{2/5}$. We can use logarithms to create a linear relationship here:

$$r = (\frac{E}{\rho})^{1/5} t^{2/5} \iff \log(r) = \frac{1}{5}\log(\frac{E}{\rho}) + \frac{2}{5}\log(t) \tag{15}$$

We can let $x = \log(t)$ and $y = \log(r)$ and then graph the related values of time and blast radius. The relationship Taylor discovered should therefore follow the linear equation:

$$y = \frac{1}{5}\log(\frac{E}{\rho}) + \frac{2}{5}x$$

Graphing the relationship between time and blast radius allows us to easily compare explosions and confirm they follow this power law. Figure (4) shows the Trinity test, and experimental data from Cristian Porneala. Showing that this relationship is not a one off, it even holds for laser beams!

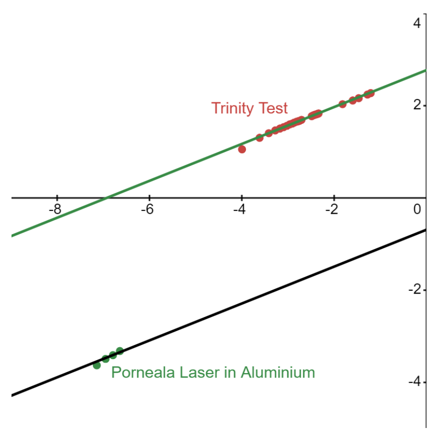

Figure 4: Two explosions with relationship between $\log(t)$ on the x-axis and $\log(r)$ on the y-axis.

Our final leap in understanding here will take us outside of our universe and into the galaxy of 'Star Wars' in order to approximate the yield of the Death Star's laser. There are a few final things we need to take into account. The density of the surrounding matter in the upper atmosphere approaching space is approximately $1 \times 10^{-5}\text{kg/m}^3$ so we will take this as our value of ρ. I have calculated a conversion from frames in 'Star Wars: a New Hope' from pixels into metres with help of the reliable source 'Wookiepedia' which claims that Alderaan (The fantastical moon in question) has a diameter of $12,500$km.

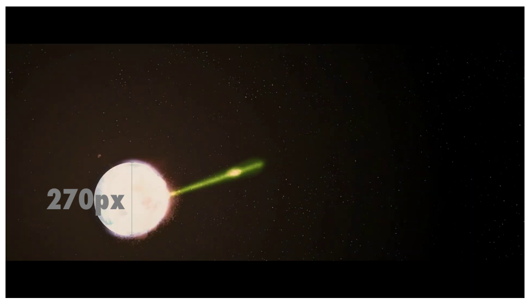

Figure 5: Alderaan moments before destruction.

Using this and subsequent frames (recorded at 30fps), I collected data points shown in the table below.

Frame	Time (s)	$\log(t)$	Radius (px)	Radius (m)	$\log(r)$
1	0.033	-1.481	133	4750000	6.68
2	0.067	-1.174	188	6714000	6.83
3	0.100	-1.00	233	8321000	6.92
4	0.133	-0.88	277	9892000	7.00

Table 2: Data from 'A New Hope'.

Using this data and 'Desmos', I graphed these points shown in Figure 6.

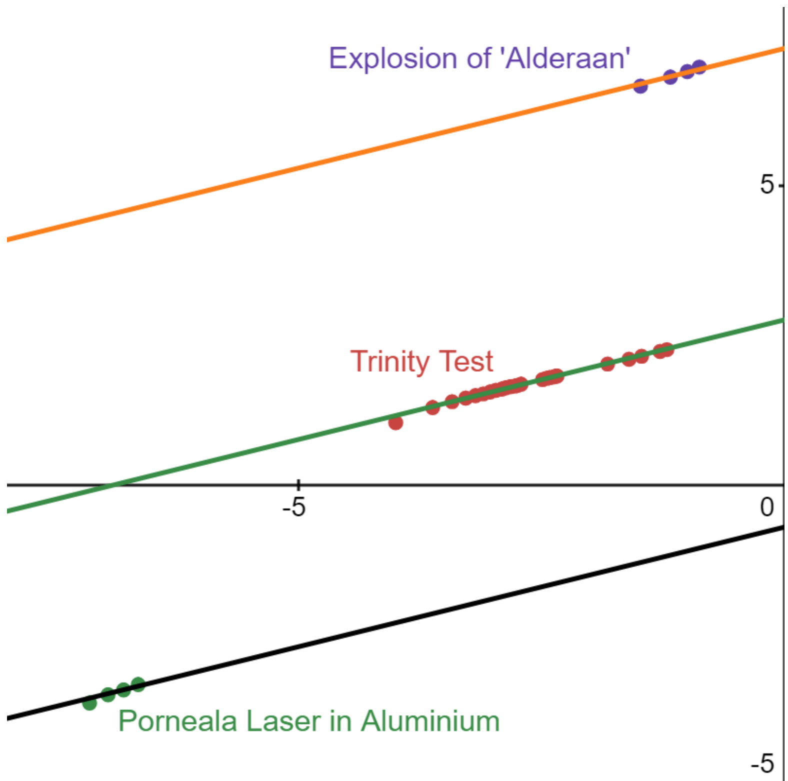

Figure 6: Explosion graphed.

We can see here that these points conform to the power law. By using least squares regression, the value of $\frac{1}{5}\log(\frac{E}{\rho})$ can be calculated as about 7.311. Finally we can compute our approximation of the energy released by the laser.

$$\frac{1}{5}\log(\frac{E}{\rho}) = 7.311 \tag{16}$$

$$E = 10^{5(7.311)} \times 10^{-5} \tag{17}$$

$$E \approx 4 \times 10^{31} \tag{18}$$

Quite a lot of energy![3] That's almost a hundred quadrillion tons of TNT!

[3] Interestingly, this is only one order of magnitude off the energy requited to overcome the gravitational binding of a similar planet (the Earth). It looks like the Star Wars technicians did their homework!

4 Units do matter.

An investigation into units whose importance is constantly (and irritatingly) enforced from such a young age reveals there importance to the maths behind physical processes, allowing us to discover and understand equations describing the world around us or even to interpret galaxies far far away. Just remember, don't forget your units!

References

[1] Strutt JW. On the Light from the Sky, its Polarization and Colour. In: Scientific Papers. Cambridge Library Collection - Mathematics. Cambridge University Press; 2009:87-103.

[2] Taylor, G. I. (1950). The Formation of a Blast Wave by a Very Intense Explosion. I. Theoretical Discussion. Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, 201(1065), 159-174.

[3] Explosions, Nature's Numbers. Available at: https://www.numbersnature.org/explosions/

[4] Cristian Porneala, David A. Willis; Observation of nanosecond laser-induced phase explosion in aluminum. Appl. Phys. Lett. 20 November 2006; 89 (21): 211121. https://doi.org/10.1063/1.2393158

[5] Lucas, G. (Director). (1977). Star Wars: Episode IV – A New Hope [Film]. Lucasfilm Ltd.

[6] Alderaan, Wookieepedia. Available at: https://starwars.fandom.com/wiki/Alderaan