<u>Fractals Don't Make Any Sense – Angus Crocker</u>

1 Introduction

Fractals are a part of maths that I had never really considered as maths at all. In comparison to the complex equations and proofs that I saw elsewhere, fractals seemed like nothing more than a fun extension of geometry. I was wrong. Fractals are a visualisation of infinity and the more I have learned, the more I have realised that fractals have exceptional implications, ones that apply to every area you could imagine.

But taking a step back, to the real world, where shapes need to have boundaries, I realised that fractals don't make any sense.

Except that they do. That's the reality of fractals, a series of counterintuitive steps that can never physically function, shapes that do not adhere to the rules of standard geometry. It is an area of mathematics that is always changing and evolving, and as such provides an interesting subject to understand. In this essay, I will introduce the idea of a fractal for an audience that may not have encountered them before, give examples for each type, explain the features that make it so interesting, and show why I believe fractal geometry to be the most exciting area of modern mathematics.

2.0.0 What is a fractal?

2.1.0 History

The term "fractal" was first used by mathematician Benoit Mandelbrot in 1975 to describe objects and shapes that could not be explained with traditional geometry. As this suggests, fractals are a very new section of maths, primarily driven by the increasing power of computer technology and the use of algorithms. In 1980 the first fractal – The Mandelbrot Set – was generated: 5 years after its original conception. It is undeniable that Mandelbrot was the pioneer of this area of mathematics, his boldness in the face of a subject that could not yet physically be processed shows his dedication to this topic.

2.2.0 Definition

Fractals are complex, never-ending patterns created by repeating mathematical equations. These come in many forms but generally fractal patterns are self-similar in nature, meaning that zooming in would result in a similar if not identical image. Fractals are inherently counterintuitive and do not function in a physical context, yet research into them is far from useless, and the ramifications of fractal research could have incredible benefits in the real world.

2.2.1 Non-Definition

The idea of fractals as only self-similar shapes is a huge oversimplification of the true depth that Mandelbrot had in mind. For him, fractals were a shift away from the idealisation of calculus as seeing the world as smooth. He wished not to create some new abstract field of mathematics, but to more firmly grasp the true nature of the world around us. He began his research with self-similar shapes, but the true idea of his research was to model a basis for the natural world.

2.3.0 Examples

Fractals are generally placed into two classifications: “additive” and “subtractive”.

2.3.1 Additive

Additive fractals are created by beginning with a simple shape and then repeatedly adding to the pattern on infinitely smaller and smaller levels. These were the first type of fractals to be modelled, and are definitely the most well-known.

Examples of Additive fractals are:

- The Mandelbrot Set
- Koch Snowflake
- Newton Fractals

2.3.2 Subtractive

Subtractive fractals are created by beginning with a simple shape such as a circle or triangle, and then repeatedly removing area by cutting out a mathematically similar shape that fills the maximum area possible.

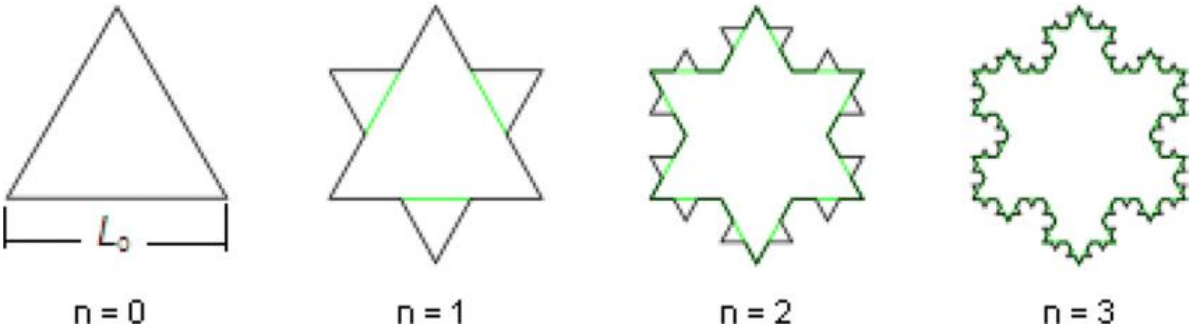
Examples of Subtractive fractals include:

- Sierpinski Gasket
- Menger Sponge
- Apollonian Gasket

3.0.0 Properties of fractals

3.1.0 Additive

For our example, we will take the Koch Snowflake, as it is a polygonal fractal.



3.1.1Perimeter

Take an equilateral triangle, side length 1. The perimeter is 1+1+1=3. On each of the sides divide the line segment by 3 and add a new equilateral triangle onto the middle third. This new shape has 12 sides of length 1/3, making the perimeter 4.

Repeating this process gives perimeters forming the sequence:

3, 4, 48/9, 192/27

To get from one term to the next, we must multiply by 4/3.

The iterative formula for this process is:

$$P = 3\left(\frac{4}{3}\right)^n$$

As n tends to infinity:

$$\lim_{n\to\infty} P = \lim_{n\to\infty} 3\left(\frac{4}{3}\right)^n$$

$$3\left(\frac{4}{3}\right)^\infty = \infty$$

∴ the perimeter of the Koch Snowflake is infinite.

Obviously, in the real world, we can recognise that this is not physically possible, however shockingly this is perhaps the least confusing property of fractals, the truly exceptional features lie ahead of us.

3.1.2 Area

Take the same equilateral triangle we began with, side length 1.

We can use the formula for the area of a triangle: $\frac{1}{2}ab\sin C$ to find that the original area will be $\frac{\sqrt{3}}{4}$.

Repeating the process of the previous calculations, we add triangles of side length $\frac{1}{3}$ to each side. The area of each new triangle will be $\frac{\sqrt{3}}{18}$, making the total area of the new shape: $3\left(\frac{\sqrt{3}}{18}\right)+\frac{\sqrt{3}}{4}$

Repeating this process leaves the sequence of areas as:

$(\frac{\sqrt{3}}{4})$, $\left(3\left(\frac{\sqrt{3}}{18}\right)+\frac{\sqrt{3}}{4}\right)$, $(12\left(\frac{\sqrt{3}}{324}\right)+3\left(\frac{\sqrt{3}}{18}\right)+\frac{\sqrt{3}}{4})$ ...

As is apparent, each new iteration of the sequence is a summation of the iterations before it, with an ever-smaller area being added on each time. This sequence can be reduced to this formula:

$$Area=\frac{\sqrt{3}s^2}{4}\left(1+\sum_{k=1}^{n}\frac{3(4)^{k-1}}{9^k}\right)$$

Where s is the original side length, k is the iteration number, and n is the limit that would leave the area of the true Koch snowflake.

This iterative formula can be explained in simpler terms as:

"At the **k**th iteration we add $3*4^{\boldsymbol{k}-1}$ more triangles of $\frac{\sqrt{3}}{4}(\frac{s}{3^k})^2$ area each. This is then added to the previous value for the area."

It is often assumed that as infinitely many triangles are being added, the area of the Koch snowflake will also be infinite. However, because each set of triangles that is being added are of smaller area than the last, the total area converges to a limit, that -whilst it cannot be reached- is an infinitely close approximation of the area. This limit is:

$$\frac{2\sqrt{3}}{5}s^2$$

As we took the original side length to be 1 unit, the final area contained within the Koch snowflake we have been using is $\frac{2\sqrt{3}}{5}$ squared units.

3.1.3 Dimensionality

One interesting feature of fractals is their breaking of the concept of dimensions. Generally, dimensions are thought of as the plane of space that an item occupies. Therefore, you may say, the Koch snowflake is clearly 2 dimensional.

However, the dimension of a fractal can also be defined using a fractal dimension, which – whilst nonsensical with real-world logic – is soundly proved with mathematical concepts.

The core concepts of fractal dimension lie in how they scale between self-similar points. For example, take a square of side length 1. This square is made of 4 mathematically similar smaller squares, each of side length 1/2.

$1/4 = (1/2)^2$ and we take the exponent as the dimension, meaning that a square is two dimensional. This concept holds true for a cube as well. It can be split into 8 cubes of 1/2 side length. $1/8 = (1/2)^3$, so a cube is 3 dimensional.

Determining the dimension of a Koch snowflake is complicated and too advanced for this introductory level essay. But a Koch snowflake is constructed of 3 Von Koch curves that are put end to end, which will have the same dimensionality as a Koch snowflake with the area removed , so this is what we will use.

Assume a van Koch curve with a length from one end to the other of 1. It can be broken into 4 similar curves of length 1/3. Using the idea that we have previously defined:

$$\frac{1}{4} = \frac{1}{3}^{x}$$

This is the same as:

$$4 = 3^x$$

By using logarithms, we can find ***x***:

$$\log_3 4 = x \approx 1.26$$

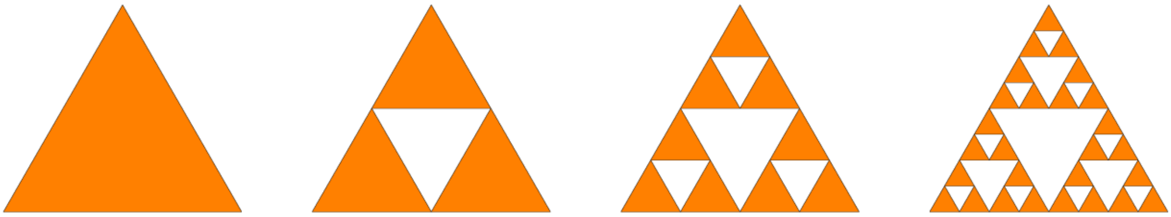
Thus, a Van Koch curve must be 1.26 dimensional.

3.1.4 Why it doesn't make sense.

The idea of non-integer dimensions obviously doesn't make sense to the average mind; however, I believe that the more perplexing feature of additive fractals is that they have an infinite perimeter that defines a finite area. When I first began research for this essay, I expected there to be contradictions, but the idea of a contained area where you could begin at one point and cover the whole surface in paint, yet you could never draw the boundary destroyed all conception of geometry for me. That is the beauty of fractals, each person can find an interest in a different one of their concept breaking nature.

3.2.0 Subtractive

As our example, we will take the Sierpinski Gasket as this is easy to construct and to understand.



3.2.1 Perimeter

Take an equilateral triangle, side length 1. The perimeter of this shape is 1+1+1=3 units. Now remove the largest inverted equilateral triangle that will fit inside the original. This new shape can be split into 3 new triangles, each with side length 1/2. The perimeter here will be 9(1/2) = 9/2. Repeat this process, 9 triangles of 1/4 side length is 27(1/4) =27/4.

The sequence of perimeters can be written:

3/1, 9/2, 27/4, 81/8, 243/16

To get from each term to the next we must multiply by 3/2. This iterative process where n is the iteration can be written:

$$P = 3\left(\frac{3}{2}\right)^{n-1}$$

As n tends to infinity:

$$\lim_{n\to\infty}(P) = \lim_{\mathrm{n}\to\infty} 3\left(\frac{3}{2}\right)^{\mathrm{n}-1}$$

$$3\left(\frac{3}{2}\right)^{\infty-1} = \infty$$

Hence the perimeter of Sierpinski Gasket is infinite.

3.2.2 Area

For this example, we will start with an equilateral triangle of area 1 square unit. For our second iteration we remove ¼ of the area, leaving an area of ¾ square units.

The sequence of areas is:

1, $\frac{3}{4}$, $\frac{9}{16}$, $\frac{27}{64}$

The iterative formula for this sequence, where n is the iteration is:

$$(\frac{3}{4})^{n-1}$$

As n approaches positive infinity:

$$\lim_{n\to\infty}\left(\frac{3}{4}\right)^{n-1} \to 0$$

As 3/4 is less than 1 the area of a true Sierpinski's triangle is 0.

3.2.3 Dimensionality

A Sierpinski Triangle is constructed of 3 identical triangles, each of side length 1/2. Using the rules that we determined in 3.1.3 it can be seen that:

$$\frac{1}{3} = \frac{1}{2}^{x}$$

$$3 = 2^{x}$$

$$\log_2 3 = x \approx 1.58$$

Therefore, a Sierpinski triangle is 1.58 dimensional, making it closer to the second dimension than the Van Koch curv

4.0.0 Further Exploration of Fractals

4.1.0 Three Dimensional?

After explaining the features and interesting properties of these fractals that lie on a flat plane, the next obvious question that arises is the possibility of (conventionally) three dimensional fractals. Indeed, these do exist and often are expanded versions of two-dimensional ones.

For example, the Menger sponge is a similar concept to the Sierpinski triangle and is constructed by continuously taking smaller and smaller cuboid sections away from a starting cube shape. They share many similar properties, although transferred to the third dimension. For example, a perfect Menger sponge has a volume of zero, yet an infinite surface area, making it the perfect theoretical shape for cost effective radio signal receivers, or as an incredibly quickly reacting solid.

Other three-dimensional fractals exist, both additive and subtractive. I find that the additive 3D fractals are mesmerising to look at and beautiful, however the utility of fractal research that makes it such an interesting subject is almost fully contained to the subtractive side.

4.2.0 Self-Similar?

To advance any further into the applications of fractal research we must move past the assumption of fractals as self-similar shapes. As a base they offer a great modelling strategy, but fractal research goes so much further.

Fractals are not inherently self-similar, and moving past this assumption allows us to explore into the natural world and apply fractals to new technology, even in a way that is not “true” fractal geometry.

4.3.0 Natural Exploration

The natural world around us contains many fractals, in imperfect forms, and the patterns that can be found allow us to calculate and understand better the world around us.

<u>4.3.1 Shells</u>

Gastropod shells are the type of shells that spiral tightly and are some of the most fascinating animal structures out there. The shell expands from a tightly coiled centre to a wide-open mouth. The sequence that defines this is called the Fibonacci sequence (also known as the golden ratio), which looks like this: 1, 1, 2, 3, 5, 8, 13, 21…. It is constructed by adding together the two previous numbers in the sequence to form the next. Despite it not being particularly intense mathematically, the Fibonacci sequence is one of nature’s most frequent fractals, appearing in shells, pinecones, and flowers.

<u>4.3.2 Ferns</u>

Ferns are the closest nature has to self-similar fractals. Inspecting a single leaf of bracken, you could see that it is constructed of many smaller copies of itself attached to a central stem. Each of these secondary sections are also made of smaller copies, if this were to be repeated infinitely, this would be a true self similar fractal. Unfortunately, this would be both physically impossible and evolutionarily impractical, so most types of ferns only have 3 to 4 iterations.

This does beg the question though, why does this shape offer an evolutionary benefit? I am no biologist, but the maximised perimeter with all the gaps must increase water runoff from the leaves, allowing water to the ground around the roots far more easily than if it were a solid leaf. The best way to do this whilst maximising surface area of the leaf for sunlight and minimising the complexity of DNA is by repeating the same structure at smaller and smaller scales. Ultimately this shows that fractals can be the simplest solution to a problem that is proposed. It also challenges my initial thoughts that the only useful fractals were subtractive, as this is clearly more of an additive example.

4.3.3 Coastlines

Coastlines of countries are where fractals can be used to understand the natural world best. When measuring a coastline, a degree of accuracy must be used. However, this can lead to inconsistencies. For example, If you were to measure the coastline of the UK with segments of 1km, the answer that you get will be smaller than if you were to use segments of 100m: the smaller the resolution of the measurement, the greater the outcome will be. This makes sense when you consider that, for every 1km segment you use, this is the shortest path between two points on the coastline. Every slight deviation from the line is a longer piece of coastline that has not been measured. Because of the inherent roughness of coastlines, this is true for every measurement that you could feasibly use. As there are always increasingly small sections, we can call this a fractal. No matter the type of fractal, the perimeter is always infinite, and therefore all coastlines must have infinite length.

However, it simultaneously cannot be true that all coastlines have the same length - the coastline of Russia is quite obviously longer than that of Belgium – fractals therefore prove the concept that some infinities are larger than others to be true. Mathematically this was already understood before fractal theory, but the real-world application makes fractals undeniably mathematically useful.

4.4.0 Fractals in the Modern World

As well as being found in nature, fractals have uses that can be implemented into human made items, influencing sections of society from art to science.

4.4.1 Chemistry

In chemical reactions a large determining factor of the rate of reaction is the surface area of any solids that are used. This is because the particles are more likely to collide and react if more surface area is in contact with other reactants. As we have previously established, a perfect 3-dimensional fractal has an infinite surface area, meaning that an approximation of it

has a very large surface area to volume ratio. For reactions that require quick completion, fractals are the best shape to use for solid components.

4.4.2 Communication

When receiving radio signals, receivers absorb radio waves. This signal is stronger if the surface area of the receiver is greater, as a higher percentage of the waves are absorbed and data corruption becomes less likely. Here the high surface area of fractals such as the Menger sponge are useful, and also saves money on the material costs, as the volume is very low. This shows how fractals can be implemented into everyday devices.

4.4.3 Architecture

Fractals also have an undeniable beauty to them, particularly ones common in nature such as the Fibonacci sequence. As a result, artist and designers often intentionally or unintentionally incorporate them into designs, particularly architecture. Such buildings include the Centenary of the federation of Australian States, and many of Antoni Gaudi's designs.

5.0.0 Conclusion

The uses of fractals perfectly sum up what they are. They are beautiful examples of geometry and are undeniably interesting to look at. Yet they also have incredible applications, both for understanding the world around us, and for new innovation in science and technology. They are a confusing intersection between art and science, and this is what makes this comparatively new field of maths one of the most interesting and potentially impactful of all. This field of mathematics is still wide open and I look forward to the new developments and applications that will inevitably keep coming in the future.