How e, i and π can make music sound better – an exploration of the relationship between music and mathematics

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Introduction

Maths and Music aren't Related… are they?

Music is a brilliant way that we can express ourselves: from the emotional climax of a beautiful symphony orchestra to the delicate yet deliberate improvisation of a jazz band, it exists all around us, and is a wonderful creative outlet that is celebrated in so many cultures around the world. To compare it with mathematics, a subject built on axioms (factual statements), seemingly restricting freedom and creativity, would seem ridiculous, yet it made perfect sense to the ancient Greeks who would intensively study music as a part of their mathematical education. Were the ancient Greeks correct about the relationship between mathematics and music? Do their ideologies hold true today? Has mathematics continued to enhance our knowledge of music? To answer these necessary questions (and more) in lovely detail, we must go back to the man who invented everybody's favourite triangle formulae...

Back to Pythagoras

Hammers and Anvils

No maths essay would be complete without the renowned philosopher Pythagoras of Samos. Born in Greece, it is believed that he realised that in any right-angled triangle, the square of the hypotenuse (the longest side of a right-angled triangle) can be calculated by the sum of the other two sides squared. However, that was not the only thing he was able to accomplish in his life. He also invented one of the first musical scales.

It is rumoured that one day Pythagoras heard a hammer strike an anvil. As he heard the note produced by that hammer, he heard another note produced by a different hammer that was twice the mass of the former. Interestingly, these two notes sounded consonant. In fact, they sounded so nice it was like they were the same note. The harmony that Pythagoras heard was an octave, and it was the start of a genius idea to invent a sufficient tuning system that could be used to improve the sound quality of music, instead of merely guessing notes by ear, which was how everyone would tune their instruments in the Ancient World.

Building a Musical Scale

Since one hammer was twice the mass of the other, one could define the notes of having a ratio of 2:1. The ratio actually refers to the frequency of the sound waves, as that determines pitch. We will explore this in more detail in Chapter 3. Since the ratio is rather simple, it is the reason why it sounds so pleasing to the ears. After 2:1, the next simplest ratio is 3:1. However, since the notes in the 2:1 can be perceived to be the same, we can use the ratio 3:2. This brings the two notes closer together and resembles a perfect fifth, which is the most pleasing harmony in music (other than the octave, of course).

Using only the octave and the fifth, a full musical scale can be created[1]. To do this, we simply start with a note, for example, A, with a frequency of 440Hz.

From here, we can work out that E must have a frequency of 660Hz since there must be a 3:2 ratio between the frequencies of the E and A sound waves. This can be achieved either by striking hammers that have a mass ratio of 3:2, or perhaps plucking strings where the ratio of their lengths is 3:2.

A fifth from E is B, though the B produced would have a frequency of 990Hz, which would be beyond our octave since an octave above A would have a frequency of 880Hz. To sort this out, we can half the frequency of the B note to 495Hz, since the octave harmony is a 2:1 ratio.

Carrying this on, we can work out the frequencies for all the other notes within the octave, and create a 7-note scale, with lovely ratios between notes. To Pythagoras, this was perfection. However, there are a few issues that came with his tuning system...

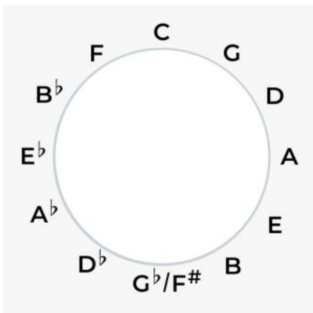
Pythagoras Problems

It was terrible. Pythagorean tuning revealed inconsistencies as musicians explored with more octaves. To see this, we need to look at the 7 notes in the scale.

Note	**A**	**B**	**C#**	**D**	**E**	**F#**	**G#**
Frequency Relative to A	1	$\frac{9}{8}$	$\frac{81}{64}$		$\frac{3}{2}$	$\frac{27}{16}$	$\frac{243}{128}$

Table 1: A table showing the notes in an A major scale and their frequencies relative to A

We have not filled in the relative frequency for D yet, so we shall do that now. We could work it out by calculating how many fifths D is from A. To do this we can analyse the circle of fifths.



[Image by hellomusictheory. Available at: https://hellomusictheory.com/learn/circle-of-fifths/]

Figure 1: The Circle of Fifths shows all 12 possible notes in the Western scale spaced apart in fifths. For a musical (major) scale, we only use 7 of these notes in a particular order. (Db = C# and Ab = G#)

[1] Reference: Shah, S (2010). *An Exploration of the Relationship between Mathematics and Music*. The University of Manchester Available at: http://eprints.ma.man.ac.uk/1548/1/covered/MIMS_ep2010_103.pdf

From here, we can see that D is 11 fifths away from A, so to work out the relative frequency of D, we can multiply 1 by $\left(\frac{3}{2}\right)^{11}$, and then divide it by 2 until it is between 1 and 2 (so it is within our octave). Doing this will give an answer of $\frac{177147}{131072}$. There is an alternative way to calculate D though, and that is considering that A is also a fifth away from D. So $y \times \frac{3}{2} = 1$, and solving for y gives us $\frac{2}{3}$, this is less than 1 so we can multiply $\frac{2}{3}$ by 2 to give $\frac{4}{3}$. Since $\frac{4}{3}$ is not equal to $\frac{177147}{131072}$, there must be something wrong with Pythagorean tuning.

The reason why Pythagorean tuning was doomed to failure is because Pythagoras made an assumption that turned out to be wrong. In order for Pythagorean tuning to work, there needs to be two integers x and y such that:

$$\left(\frac{3}{2}\right)^x = 2^y$$

Rearranging this we can find that:

$$x \log_2\left(\frac{3}{2}\right) = y$$

And dividing both sides by x reveals that:

$$\log_2\left(\frac{3}{2}\right) = \frac{y}{x}$$

Since $\log_2\left(\frac{3}{2}\right)$ is irrational, it means that there are no integers that can solve this equation. When x is 12 and y is 7, it nearly works, you simply have to assume that 129.7463379...=128. Known as the Pythagorean comma, this false assumption meant that Pythagorean tuning was a flawed system. Was there a better way to tune notes though?

Creating a Compromise

Pythagorean tuning may have been a flop, but it was not a massive flop. For instance, Pythagoras was still correct about the idea that a fifth resembled a 3:2 ratio and an octave 2:1. But instead of aiming to preserve perfect ratios, we could evenly space all 12 notes within the octave, so that the relative frequencies between two consecutive notes remains the same. This means that we need a value for x such that:

$$x^{12} = 2$$

When we solve for x, we get that:

$$x = \sqrt[12]{2}$$

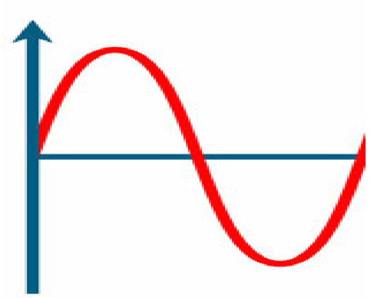
This means if we know the frequency of 1 note, we can work out the frequency of the note next to it by multiplying by $\sqrt[12]{2}$. We tempered with the frequencies slightly to create notes with equal frequency ratios. This method of tuning is known as equal temperament. Invented by Simon Stevin[2], it is the tuning system that mainly used even in the modern day. However, it would mean that the ratio of a fifth would now be 2.9966:2, which is very close to 3:2 but not quite. In fact, it is close enough to the point that our ears do not really care that it is not a perfect ratio.

So, as we can see, mathematics has had great influence on how we are able to tune our instruments in the modern day, but has mathematics had a deeper influence on the music we listen to today?

A Deeper look into Sound Waves

Sine Waves and Sound

In 1822, Joseph Fourier's work on conducting heat in solid bodies consequently proved that any sound can be graphed through a combination of sine and cosine waves. To see how this works, a simple wave is shown below:



[Image by Zizo. Available at: https://stock.adobe.com/uk/search?k=sine+wave+graph&asset_id=508442743]

Figure 2: Sound can be modelled with a sine wave, where frequency is the pitch and the amplitude is the loudness of the sound.

This graph can be modelled with $y = a\sin(bx)$, where a determines the amplitude (height of the peak of the graph) and b determines the frequency of the sound (number of times the wave passes a point per second). The amplitude of the sound wave tells us how loud the sound wave will be, and, as we saw in Chapter 2, the frequency of the sound wave determines the pitch of the sound produced.

Even though a sound can be modelled with a sine wave, this merely represents the "purest" version of what a note can sound like. In reality, most sounds that exist in nature are composed of a combination of these sine waves, each bearing different amplitudes and frequencies. This includes the plucking of the A string on a guitar and the playing of the A on a piano – but – if they are made of so many different sin

[2] Reference: Lugg, O. (2021). *How Pythagoras Broke Music (and how we kind of fixed it)*. [online] www.youtube.com. Available at: https://www.youtube.com/watch?v=EdYzqLgMmgk.

waves stacked together, how do they sound so beautiful? And how can both instruments play the same note yet sound so different?

The Harmonic Series

Imagine that we have all the natural numbers in a list of ascending order, and we wanted to find the sum of them. It would look something like this:

$$1+2+3+4+5+6\ldots$$

If we were to instead find the sum of the reciprocals of each of these numbers, we would instead have a sum like this:

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\ldots$$

This series of adding up numbers is known as the harmonic series, and due to the lackadaisical nature of mathematicians, it is often written like this:

$$\sum_{r=1}^{n}\frac{1}{n}$$

where n is the number of terms in the series that we want to add together. Rightfully named the harmonic series, this series helps us to further understand how musical instruments work.

Notes on a Guitar String

When a guitar string is plucked, a wave is produced that is reflected off the other end. This creates the sound known as the fundamental, which determines the pitch of the note. Because it is reflected, it appears as if only half of the wave is there, so if we call the length of the string l, the wavelength is $\frac{1}{2}l$. From now on, we will call the fundamental wavelength w.

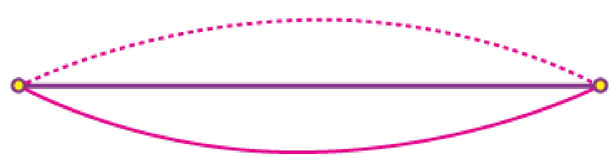
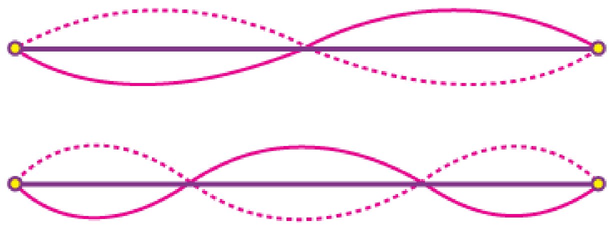


Figure 3: The fundamental wave formed when a string is plucked. This can also be formed by blowing into a flute, as the wave reflects off the walls of the instrument.

As the wave reflects, it interferes with itself in a phenomenon known as superposition, producing other waves:



[Images Fig 3 and 4 by BYJU'S. Available at: https://byjus.com/physics/overtones/]

Figure 4: These are the first two harmonics produced by a wave (yes, they are to scale with Figure 3)!

If we look at wavelengths of the next two waves, we see that they are $\frac{w}{2}$ and $\frac{w}{3}$ respectively. These waves produce sounds that are known as harmonics. They are not as easy to hear as the fundamental, but they do help in creating the sound that is heard.

If we add up these wavelengths, including the fundamental, we get:

$$w + \frac{w}{2} + \frac{w}{3}$$

Taking out w as a factor shows us that the sum can be written as:

$$w\left(1 + \frac{1}{2} + \frac{1}{3}\right)$$

Notice that inside the brackets, we have the first 3 terms of the harmonic series! This is because when a wave superposes along a string, the other waves produced always have a wavelength that is an integer divisor of the fundamental wavelength. And since the harmonic series is a divergent series, an infinite number of these waves can be produced. The amplitudes of these waves do vary, which can make different notes sound different, and is how a piano can be distinguished from a guitar or a flute from a trumpet.

With the information that we have about the wavelengths, we can also work out the frequencies, and therefore pitches of each harmonic, if we know the frequency of the fundamental using the formula:

$$v = f\lambda$$

Where v is the speed of the wave, f is the frequency and λ is the wavelength of each wave. Since the waves are all from the same string, v is constant, which means:

$$\lambda \propto \frac{1}{f}$$

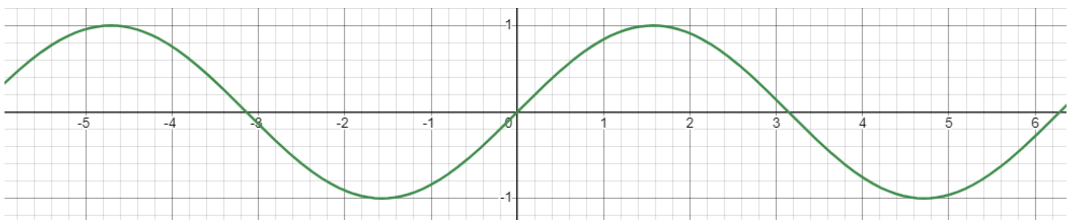
This tells us in the way that the wavelength of each successive harmonic follows the harmonic scale, the frequencies of these harmonics will always be integer multiples of the fundamental frequency. For example, if the fundamental frequency was 440Hz, the following harmonics would be 880Hz, 1320Hz, 5280Hz etc.

So, the harmonic series is crucial in knowing how different frequencies interact in the notes that we play, but Fourier had a lot more to say about the power that sine waves hold in sound production...

Maths and Music in the Modern Day

Approximating Sine

Despite the fact it is incredibly useful that sound waves can be modelled with a sine graph there are many more useful traits associated with sin and cos that help develop the music we listen to. Consider a sine graph:



[Image created using Desmos. Available at: https://www.desmos.com/calculator]

Figure 5: The graph f(x)=sin x.

To approximate the shape of this graph, we could use an infinitely long polynomial, and we can calculate this using the Taylor Series Expansion which states that:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

[Reference: Wikipedia Contributors (2019). *Taylor series*. [online] Wikipedia. Available at: https://en.wikipedia.org/wiki/Taylor_series.]

Where $f^{(n)}(0)$ is the value of the n^{th} derivative when $x = 0$. In our case, we will make $a = 0$, so:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

This is known as a Mclaurin series. To make things simpler, let us only consider the first 4 terms of the series:

$$\sum_{n=0}^{3} \frac{f^{(n)}(0)}{n!}x^n = \frac{f(0)}{0!}x^0 + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

We want $f(x)$ to equal $\sin x$, so we can now substitute values for the numerators.

For the 0th term, we need to calculate $f(0)$ when $f(x) = \sin x$. Since $\sin 0 = 0$, it means that the 0th term is simply equal to 0.

Now we consider the 1st term. When $f(x) = \sin x$, $f'(x) = \cos x$. We want $f'(0)$, which is $\cos 0$, and that would be equal to 1. This is our first term:

$$\frac{1}{1!}x$$

Which simplifies to x .

Looking at the second term, we need to work out $f''(0)$. This is equal to $-\sin 0$ which is 0. This means the second term is 0. When we look at the 3rd term, $f'''(0)$ is equal to $-\cos(0)$ which is equal to –1.

So our third term looks like this:

$$-\frac{x^3}{3}$$

Therefore, the first four terms of the series are as follows:

$$x - \frac{x^3}{3!}$$

If we carry on this method, instead using the first 11 terms, we get this sum:

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}$$

We can now see that a pattern starts to emerge. Firstly, only the odd powers of x emerge, which means we can write each x value in the form x^{2r+1}, (where r is an integer greater than or equal to 0). The denominator is always equal to the exponent factorial, so that can also be written as $(2r + 1)$!. The sum also has alternating "plus" and "minus" signs, which we can represent with $(-1)^r$, since –1 to the power of an odd integer is always negative and to the power of an even number it is always positive. With this information, we can conclude that:

$$\sin x = \sum_{r=0}^{\infty} \frac{x^{2r+1}(-1)^r}{(2r+1)!}$$

[Reference: blackpenredpen (2018). *Power series of sin(x) and cos(x) at 0*. *YouTube*. Available at: https://www.youtube.com/watch?v=zgwUMJOXwDM.]

This now means we have an infinite series to calculate sin! Using the same method, we can also calculate an infinite series for cosine:

$$\cos x = \sum_{r=0}^{\infty} \frac{x^{2r}(-1)^r}{(2r)!}$$

[Reference: blackpenredpen (2018). *Power series of sin(x) and cos(x) at 0*. *YouTube*. Available at: https://www.youtube.com/watch?v=zgwUMJOXwDM.]

So, as we can see, cosine contains all even powers of x , while sine has all the odd powers. What if we were to add them together?

This is what prodigious mathematician Euler thought, and through this, he came up with one of the most important formulas in the study of complex numbers.

The creation of Euler's Formula

If we multiply the sine expansion by i , and add it to our cosine expansion, we get:

$$\sum_{r=0}^{\infty}\frac{x^{2r}(-1)^r}{(2r)!}+i\sum_{r=0}^{\infty}\frac{x^{2r+1}(-1)^r}{(2r+1)!}$$

Having i in our sum makes things more awkward, so to try to simplify things, we could rewrite -1 as i^2:

$$\sum_{r=0}^{\infty}\frac{x^{2r}(i^2)^r}{(2r)!}+i\sum_{r=0}^{\infty}\frac{x^{2r+1}(i^2)^r}{(2r+1)!}$$

When we expand the brackets on the numerator, we get:

$$\sum_{r=0}^{\infty}\frac{x^{2r}\cdot i^{2r}}{(2r)!}+i\sum_{r=0}^{\infty}\frac{x^{2r+1}\cdot i^{2r}}{(2r+1)!}$$

We can then bring the i in the sine expansion to the numerator, and simplify both numerators to get:

$$\sum_{r=0}^{\infty}\frac{(ix)^{2r}}{(2r)!}+\sum_{r=0}^{\infty}\frac{(ix)^{2r+1}}{(2r+1)!}$$

Which can just be written as:

$$\sum_{r=0}^{\infty}\frac{(ix)^{2n}}{(2r)!}+\frac{(ix)^{2r+1}}{(2r+1)!}$$

With this new simplified expression, combining the cosine and sine series, the first few terms look like as follows:

$$1+ix+\frac{(ix)^2}{2!}+\frac{(ix)^3}{3!}+\frac{(ix)^4}{4!}+\frac{(ix)^5}{5!}+\frac{(ix)^6}{6!}\dots$$

Interestingly, this new series looks incredibly similar to another infinite series: e^y :

$$e^y=1+y+\frac{y^2}{2!}+\frac{y^3}{3!}+\frac{y^4}{4!}+\frac{y^5}{5!}+\frac{y^6}{6!}\dots$$

So, if we substitute y for ix , like Euler, we can now prove that[3]:

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[3] Reference: qncubed3. (2019). *Euler's Formula Using Taylor Series Expansions. YouTube.* Available at: https://www.youtube.com/watch?v=OSGh8ZEFDJE

$$e^{ix} = \cos x + i \sin x$$

This formula is very useful as x can tell us how far we have travelled along a wave in radians, and this is crucial to understanding how the Fourier Transform works.

What is the Fourier transform?

The Fourier Transform is a formula that is very useful in many areas of music, such as autotune, removing unwanted frequencies, identifying instruments, the list goes on! It looks a little somewhat like this:

$$\hat{g}(f) = \int_{-\infty}^{\infty} g(t) e^{-2\pi i f t} \, dt$$

[Reference: 3Blue1Brown (2018). *But what is the Fourier Transform? A visual introduction. YouTube*. Available at: https://www.youtube.com/watch?v=spUNpyF58BY.]

It looks rather complicated, but for the number of applications it possesses, it looks rather simple and elegant. We have modelled our sound waves with sine graphs though, how would e, i and π correlate to music?

Notice the $e^{-2\pi i}$. This is essentially telling us that we are considering -2π radians (360 degrees clockwise) of the wave. We also multiply this by t (time), as that can measure for us the amount of time taken to go through one cycle. But perhaps the wave oscillates multiple times per second, so, to consider this, we multiply the exponent by f (frequency).

We also multiply $e^{-2\pi i f t}$ by g(t) as g(t) represents our wave function, so that we can consider the position of time and frequency against our wave. Finally, for each point, we take an integral, to consider the limits as we consider every position on our wave function. Doing all this gives us ĝ(f), which tells us the frequencies of the various sine and cosine waves that make up g(t), and their respective amplitudes.

With this formula, we can essentially examine any sound wave, and analyse it to identify frequencies and amplitudes associated with other instruments and can thus be used by computers to distinguish between a piano and a guitar for example. However, the Fourier transform is a topic that extends well beyond sound and music, and can be used in quantum physics, the Riemann Zeta function and many more areas of maths and science. It is definitely something worth looking at in more detail!

Conclusion

In conclusion, there is so much evidence to suggest that mathematics and music share an intimate relationship, despite looking like polar opposites on the surface. From the Ancient Greeks to modern day technology, music crops through so many areas of maths, including trigonometry, infinite series and even the study of complex numbers. It seems that mathematics continues to be the underlying structure to which music relies on. Or is music the relentless melody that perpetually keeps the rhythm of mathematics in motion? I guess that is a question for you to answer for yourself!