Knot Your Ordinary Game Show

Imagine you find yourself in a game show with the chance to win $10,000,000. You will be locked in a room and given various tasks to complete, but the game show host, being an enthusiastic fan of knot theory, will only ask you questions regarding knots. This is how I will introduce some knot invariants and how you might 'invent' them yourselves, given that you have an uncanny aptitude for coming up with and proving knot invariants, of course.

What is a knot, you might ask? Mathematically, it's defined as a circle embedded in $\mathbb{R}^3$ by the function $f: S^1 \to \mathbb{R}^3$, or, more informally, a piece of string that might be twisted or tied in some way with both ends joined together.

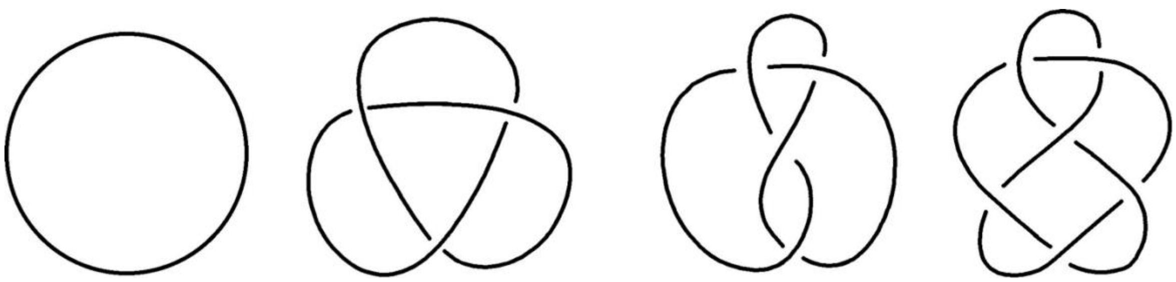
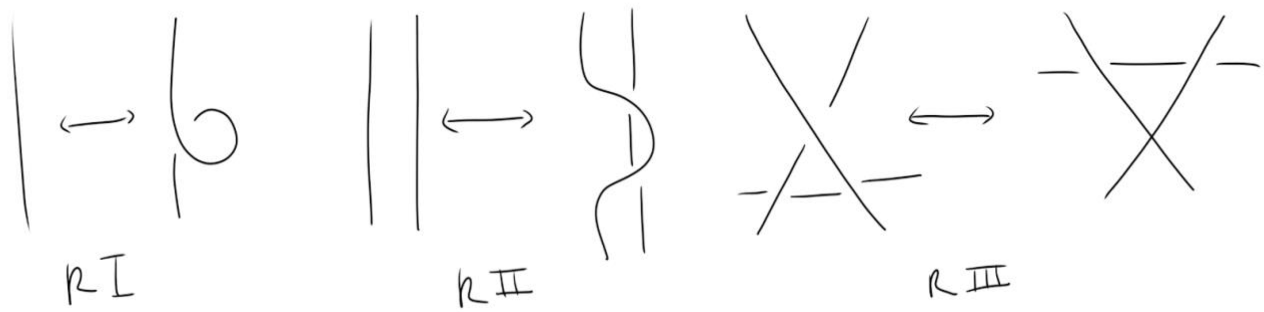


Figure 1: Examples of knots

Knot theory is one of the newer fields of maths, coming into existence in the 1800s at the same time as other fields of maths such as Galois theory and differential geometry. Gauss was one of the first people to mathematically formalise knots, and also did some work with links (which are two or more knots joined together). He managed to figure out that to find how 'linked' two links are, you can calculate a certain double integral, where γ_1, γ_2 are different links:

$$\begin{aligned} \text{link}(\gamma_1, \gamma_2) &= \frac{1}{4\pi} \oint_{\gamma_1} \oint_{\gamma_2} \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \cdot (d\mathbf{r}_1 \times d\mathbf{r}_2) \\ &= \frac{1}{4\pi} \int_{S^1 \times S^1} \frac{\det(\dot{\gamma}_1(s), \dot{\gamma}_2(t), \gamma_1(s) - \gamma_2(t))}{|\gamma_1(s) - \gamma_2(t)|^3} \, ds \, dt \end{aligned}$$

However, this essay will neither deal with links (mostly) nor with monster integrals such as the one above, and will focus mainly on knots and invariants. When studying knots, we are interested in whether two knots are equivalent, i.e. if one of them can be manipulated in some way so that it becomes the same as the other, as this allows us to classify all the different knots that can exist. All of the possible ways a knot can be manipulated are called the Reidemeister moves, which are as follows:



We can tell knots apart from knots that are fundamentally different to them (i.e. where you cannot get from one knot to the other via the Reidemeister moves) with different invariants. Invariants are properties that stay the same for all representations of a knot, and vary in strength. The strength of an invariant signifies how successful they are at telling different knots apart – more information will follow later on.

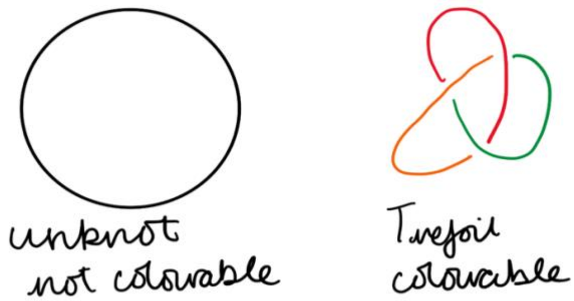
TASK I

You walk into a room with a desk, an assortment of pens on the left and a stack of paper at its foot. Twin speakers stand in the far corners of the room, with a sizable TV screen propped up in front of the desk. The door shuts and a voice comes from the speaker:

"This will be your first task. Since all the tasks will be related to knot invariants I'll first introduce you to the most simple of the knot invariants, colourability. There are two rules for colourability.

1. *At each crossing (a point in a knot where one arc passes over another – for example there are three crossings in the trefoil, the second knot in Figure 1), you must either use three different colours or only one colour.*

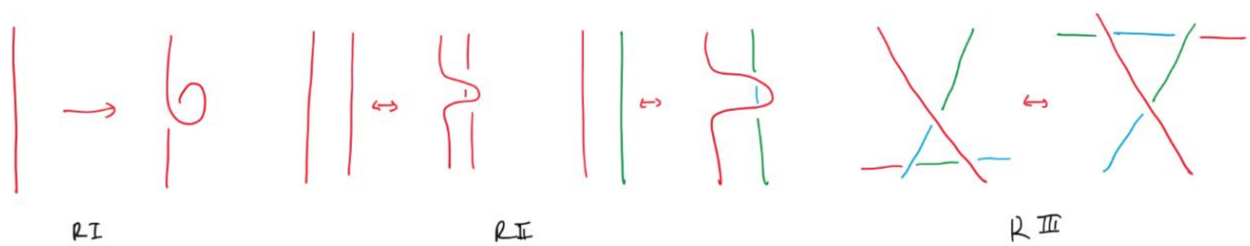
2. *When colouring a knot, you must use at least 2 colours.*



The first task will be on the colourability of a knot. I hope you have your coloured pens ready."

An aside on the invariability of colourability

What the game show host failed to prove in their eagerness to introduce colourability is that it is indeed an invariant. To do this we essentially show that colourability is preserved under all 3 Reidemeister moves:



There are actually a few more cases for RIII, but these involve the same colours coming in and out of a crossing and so we can see that all the Reidemeister moves preserve colourability, and it is indeed an invariant. Aside over.

A trefoil appears on the screen. After making full use of the coloured pens, you convince yourself that it is 3-colourable. A vague memory of a course you took a while back on knot theory surfaces: knots can be classified with p-colourability, where p is some prime number and represents the prime number of colours needed to colour the knot (although not all of them need be used). The voice interrupts:

"Good! The trefoil is indeed 3-colourable. I wonder how you would fare without your coloured pens, though? Our viewers would love to find out!"

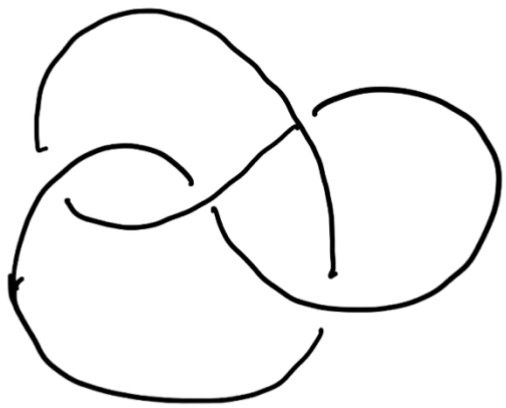
TASK I COMPLETE

TASK II

A compartment in the desk hisses open and the coloured pens promptly fall in one by one.

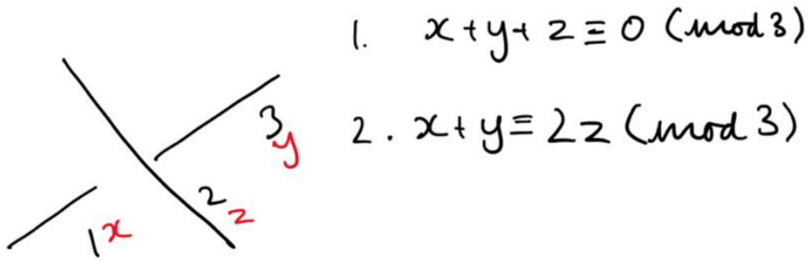
"Here is your next task: Find this knot's p-colourability. Good luck!"

A knot appears on the screen, completely different to the trefoil that was displayed previously.



Refusing to be thwarted by a mere lack of coloured pens, you do some testing. Instead of colours, you decide to use numbers to represent the colours, and experiment a little with modular arithmetic to get a relationship between the arc that goes over and the 2 arcs that pass under at a crossing.

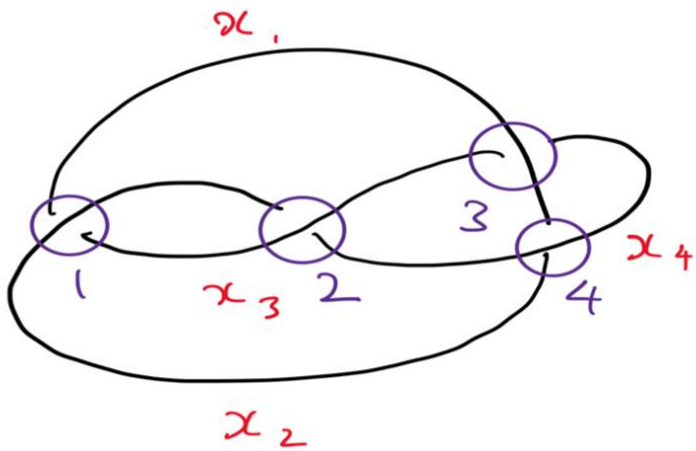
An arc or a strand in a knot will refer to an unbroken section of a knot, a continuous line that ends when it is cut off by crossing under a different arc.



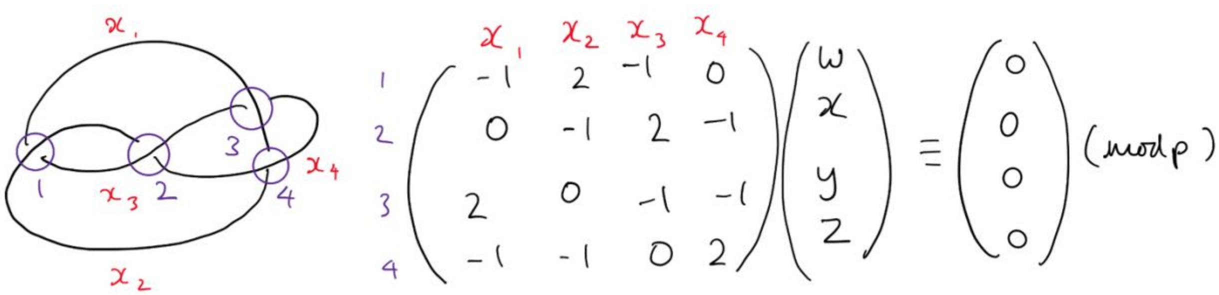
After doing more testing, relationship 1 becomes more and more attractive for rejection. After thinking about it for a while, it becomes obvious that it does not generalise to higher p-colourability, i.e. the equation $x + y + z \equiv\ 0\ (mod\ p)$ (where p is some prime number) is only true for $p = 3$. This is easily shown by a counterexample: Let $x = 1, y = 3, z = 2$ as above. If we let p be some prime, say

5, then $x + y + z \equiv 1\ (mod\ 5)$. However, with the second relation, you get that $1 + 3 \equiv 2 \times 2\ (mod\ 5)$, which still holds. Therefore, you decide to take the second relationship as fact.

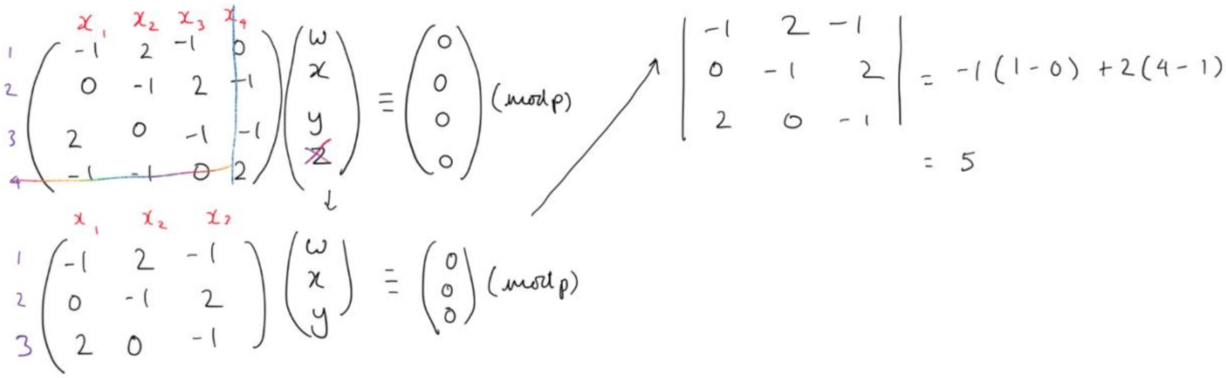
Armed with the second relationship, you copy down the knot on the screen and label the separate arcs with x_1, x_2, x_3, x_4, and all the crossings with $1, 2, 3$ and 4.



You manipulate the second relation into the form $2z - x - y \equiv 0\ (mod\ p)$. With delight, you come to the realisation that since you have 4 crossings that must obey that relation, you can form a 4x4 matrix to represent the whole system of equations, labelling each column with x_1, x_2, x_3, x_4 and each row with each of the crossings, $1, 2, 3, 4$, and at each crossing labelling the over-crossing with 2 and each of the under-crossings with -1.



You spot that one of the equations is linearly dependent on the other ones, and hence does not need to be in the matrix. This results in 3 equations and 4 variables, but one of the variables can be expressed in terms of the other ones so you can also delete one of the variables. And now you are stuck. What more can you do with this? Panicking slightly, you decide to calculate the determinant of the resulting matrix on a whim. You reach into your pockets for your saviour: The graphical calculator, a behemoth that can compute up to 10x10 matrix determinants and inverses and a calculator that would fit in no pocket other than one fabricated for the purpose of the story.

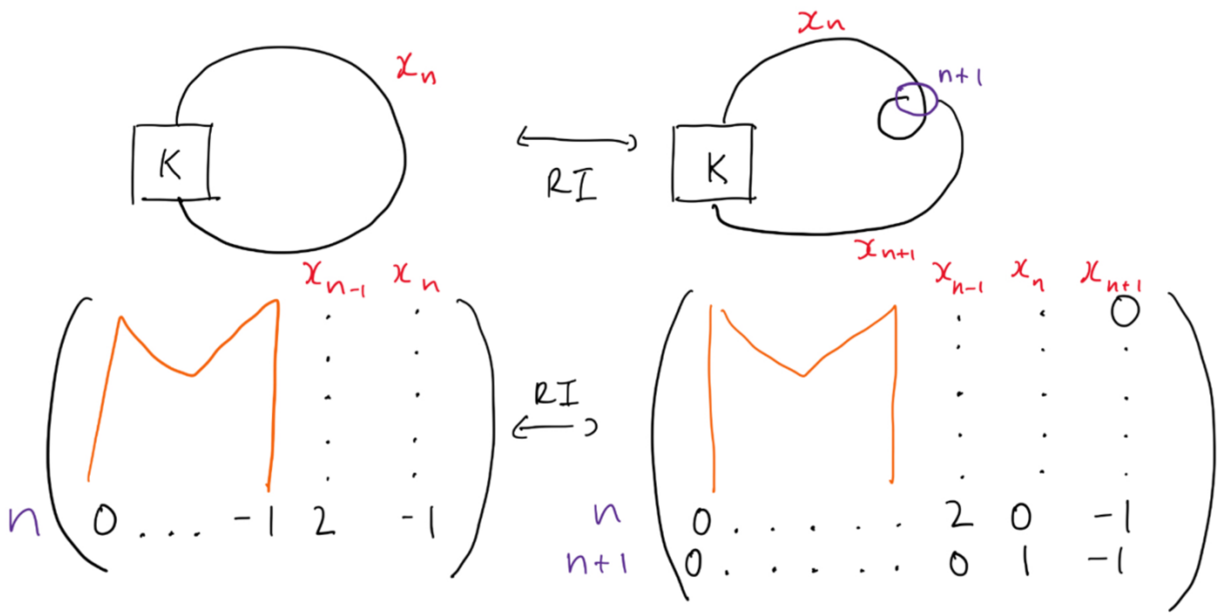


After punching in the numbers, your calculator spits out 5. In a stroke of genius, you realise something: Usually, you want the determinant of the matrix to be non-zero, as this means that there are unique solutions to the system of equations that the matrix represents due to existence of an inverse matrix. This is what you have here, a matrix with determinant 5. However, if you think of w, x, y, z being the different colours used to colour a knot, the determinant being non-zero is a problem, as it implies that there is one and only one way to colour the knot. But of course this is not true! You can always switch around the order of the colours to get a different colouring of the matrix, and so you strangely *want* the determinant of the matrix to be zero so that there can be infinitely many solutions to the system of equations. And here you remember: You are working mod p. By choosing a prime number p that is a factor of the determinant, you can *force* the determinant of the knot to be zero, and from here you get the relationship $p \mid \det(M)$, where M is the matrix that you have deleted the rows and columns from!

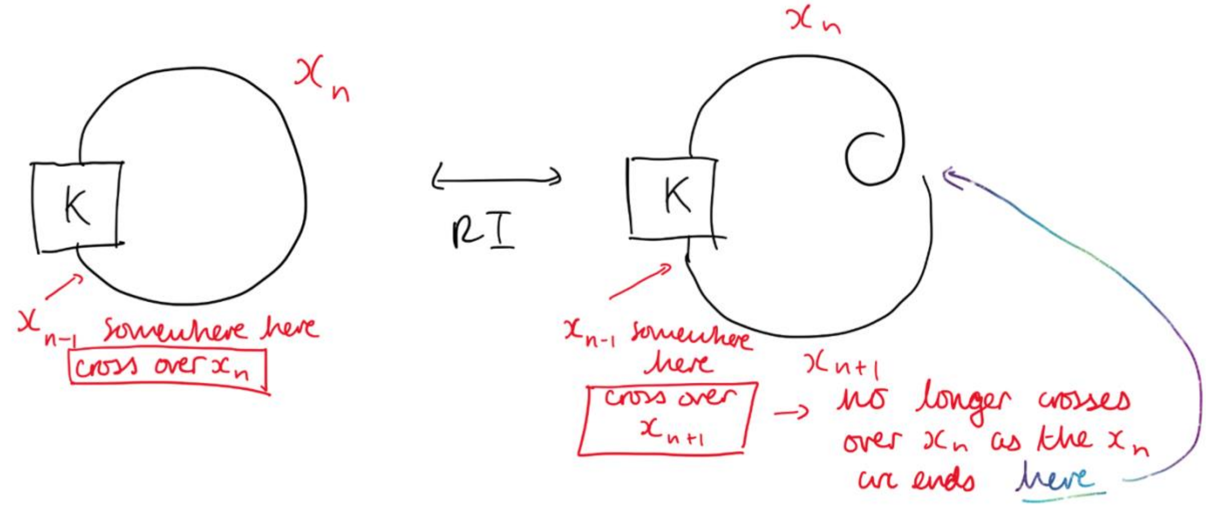
Hence you have figured out that the figure 8 knot is 5-colourable. You realise that as long as you can calculate the determinant of the resulting matrix, you can determine the colourability of a knot, and hence tell them apart with (relative) ease using the invariant of the p-colourability of a knot. If a knot is 3-colourable and another knot is 5-colourable, you can safely conclude that they are fundamentally different knots! (The invariant of colourability is only helpful up to a point, however, as will be demonstrated shortly.)

Another aside on the invariability of the knot matrix

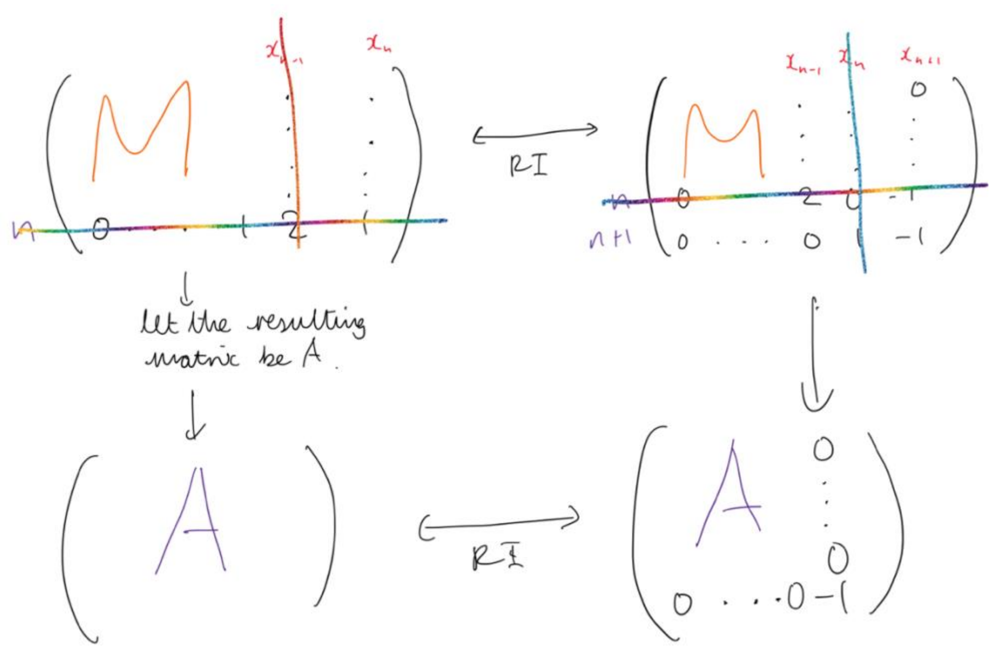
Similarly to the colourability of a knot, we can also just prove that all of the Reidemeister moves preserve the determinant of the resulting knot matrix. Also note that this matrix must always be square, as you can probably convince yourself that the number of crossings = the number of separate arcs in a knot. This is more complicated and long winded to prove than the colourability of a knot, so we will just prove that it stays invariant for RI. We draw a diagram of some knot with n crossings with a strand x_n on the end, and then we apply RI to the knot to introduce a new crossing, $n+1$, which increases the number of columns and rows of the matrix by 1 each.



We imagine that x_{n-1} is somewhere in the knot in the bottom half of the arc x_n and crosses over it at some point, hence the value of 2 in the matrix on the left. When the arc x_{n+1} is introduced, x_n no longer crosses over x_{n-1} at the previous point and the value is hence 0 in the new matrix. The 1 in the last row is because x_n is both the over-crossing and the under-crossing at that point, hence the entry in the matrix is $2-1=1$.



When we choose specific rows and columns to delete, specifically the n row and x_{n-1} column in the left matrix and the n row and x_n column in the right matrix in this case (shown below). What we get as a result is the same matrix on the left and the right, but we have a -1 in the bottom right corner as the x_{n+1} strand is the under-crossing in the new crossing $n + 1$ that we introduced via RI, which will give us the same magnitude of determinant in both cases, but the matrix of the knot after RI will be negative.



We can see that we should really have defined the determinant of the knot as $|\det(M)|$, as there is the possibility that the resulting determinant is negative. We can do similar things with the other Reidemeister moves, and so we can prove that this is invariant. Complicated matrix aside over.

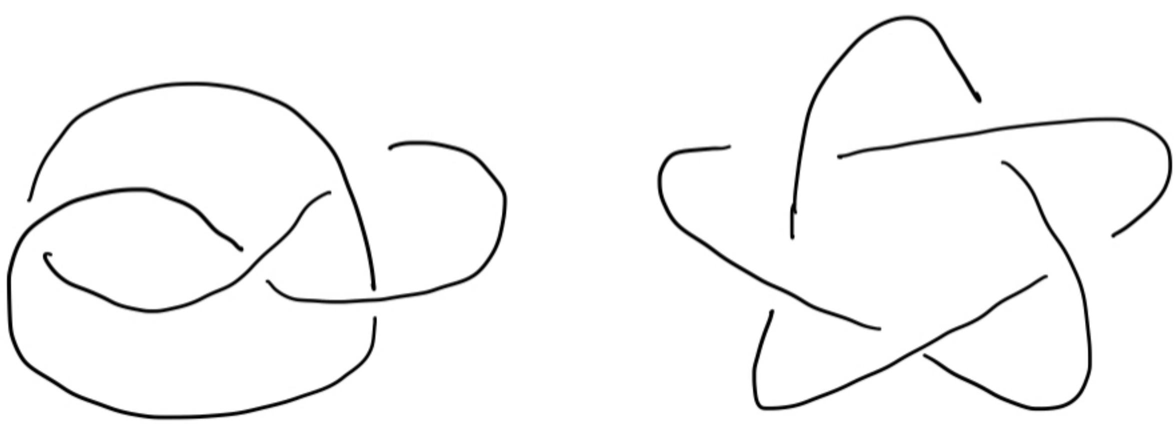
You take a new sheet of paper and scribble the number 5 onto it, and hold it facing towards the screen. A voice explodes from the speaker:

“Correct! The figure 8 knot is indeed 5-colourable.

TASK II COMPLETE

TASK III

Your next task is to prove that these two knots are fundamentally different:



Key word ‘prove’. An attempt at showing that one knot could not be morphed into the other one would not only be time consuming, but would also lose you your chance at $10,000,000.”

You scoff at the triviality of this problem, armed with your new method to find the determinant of a knot and hence determine its colourability. You confidently follow the algorithm to find the matrix you need to take the determinant of and input the matrices into your calculator. You find out that the determinant of the figure 8 knot (left) is 5, and the determinant of the cinquefoil (right) is 5. What can we do from here? The determinants of the knots are both the same, so the invariant of colourability does not help us here.

You wrack your mind, trying to figure out how to prove that they are different. (This is a problem with the p-colourability of a knot as an invariant – there are lots of knots with the same determinant and hence colourability, and so you can’t classify them as different without some other invariant.)

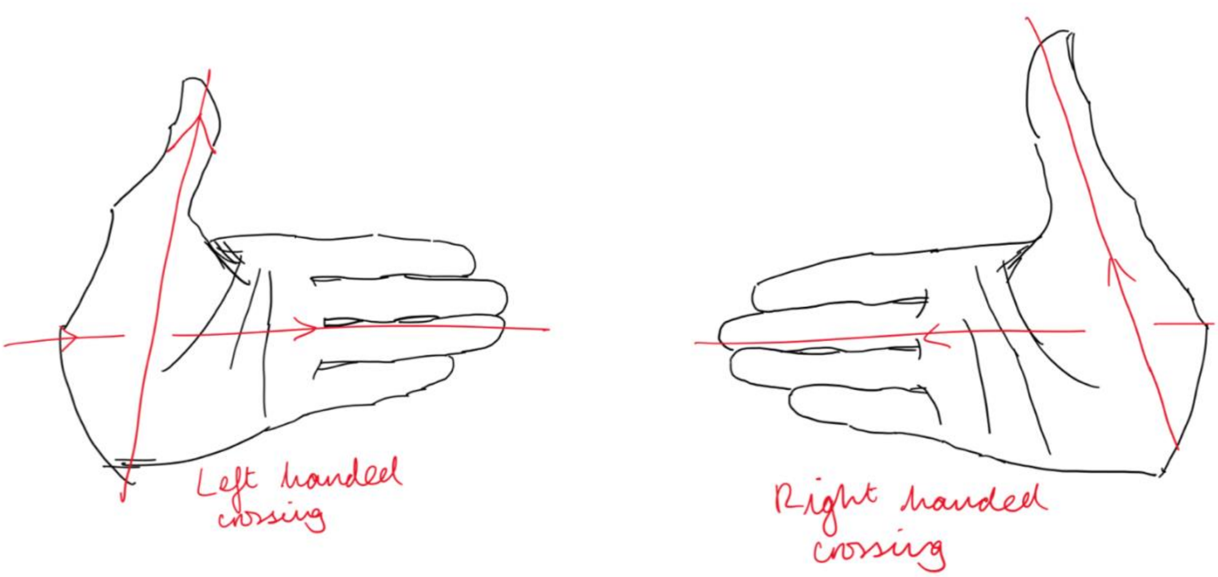
"This is the last task. You still have a hint – I would advise that you use it."

"Alright. I would like to use my hint, please."

"I would advise that you consider assigning a direction to the knots, which would give them some orientation, and then thinking about left and right-handed crossings and how you could simplify the knot."

You give some orientation to the knots as follows:

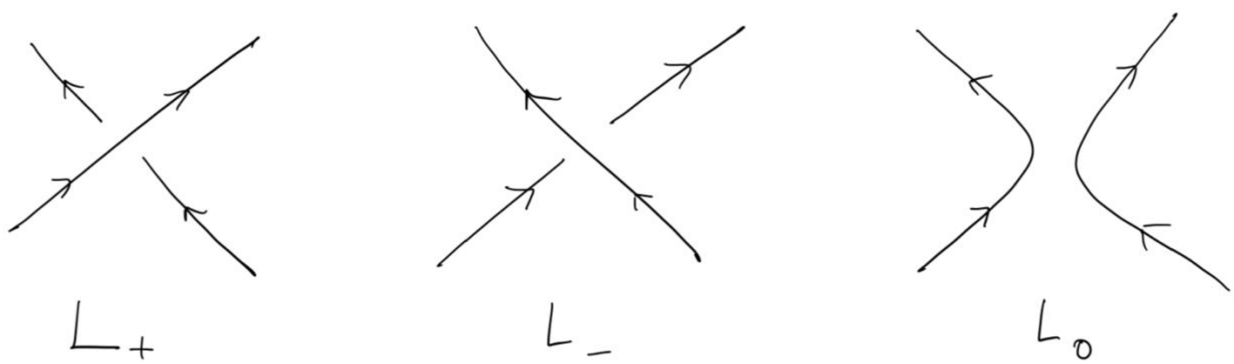
Then you ponder the problem of left and right-handed crossings for a while. In the end, you decide to define a right-handed crossing as a crossing in which you hold your thumb and fingers at right angles with your palm facing up and you point the thumb of your right hand in the direction the over-crossing, and point your fingers in the direction of the under-crossing, and vice versa for a left-handed crossing. This is easier explained in a diagram:



For simplifying the knot somehow, you might consider somehow ‘undoing’ some of the crossings. This might include turning right-handed crossings into left-handed ones by ‘pulling’ the under-crossing on top, or undoing crossings entirely by taking 2 different arcs going in the same direction and joining them up, so that the crossing never existed (more clearly shown below). So you decide on this method:

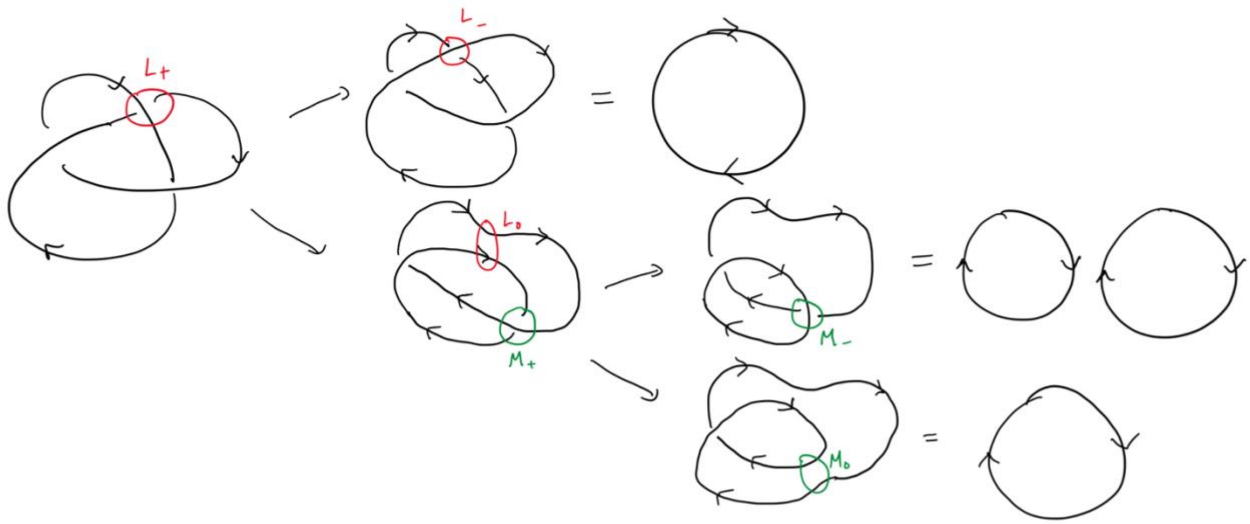
1. Take a crossing
2. Figure out its orientation (i.e. if it’s left-handed or right-handed)
3. Split the knot up into two parts:
    a. The knot if the orientation of the crossing was reversed (i.e. change LH to RH and RH to LH)
    b. The knot if the crossing didn’t exist at all (while maintaining the direction of the knot – this is important)
4. Continue until the knot is fully decomposed into a combination of unknots and unlinks.

What is being described is generally known as a skein relation, and, unbeknownst to you, is a method used to recursively come up with polynomial invariants for knots, such as the Jones polynomial and the Conway polynomial, the latter of which you will imminently ‘invent’.



Here, L_+ indicates right-handed crossing, L_- indicates left-handed crossing and L_0 indicates no crossing at all.

As it turns out, you have found a method to successfully 'simplify' the knots so that you either end up with the unknkot (just the circle) or the unlink (just two circles)! Animatedly, you try decomposing a trefoil and end up with two unknots and one unlink:



Using this newfound method of simplifying knots, you decide that you want some sort of polynomial invariant, which you decide to confusingly denote with the exact same symbol as the grad vector, ∇.

After a little thought, you contemplate the thought of defining $\nabla(unknot)$ as 1 and $\nabla(unlink)$ as 0, as those are what the knot can ultimately be decomposed into, and putting some letter z in the equation somewhere so that ultimately a polynomial can be reached and the output might be a knot invariant.

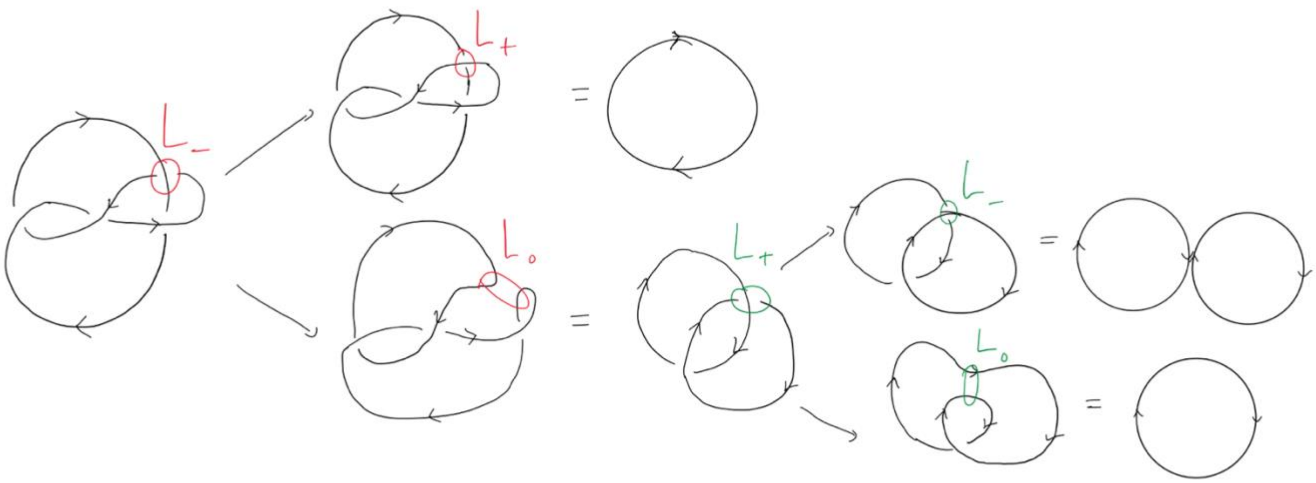
There can be both left handed and right handed crossings that you might choose to start decomposing the knot from, but realistically you would never choose to look at a part of the knot where two arcs do not meet and introduce a new crossing, as this would achieve the opposite of your aim of simplifying the knot. Therefore, working through it logically, you would want to have the variable z separate to the left and right-handed crossing terms in the equation and stick it onto the L_0 term. Having the z variable in the denominator could be an issue too, as dividing by z could cause two knots to ultimately have the same polynomial which diminishes the power of the invariant.

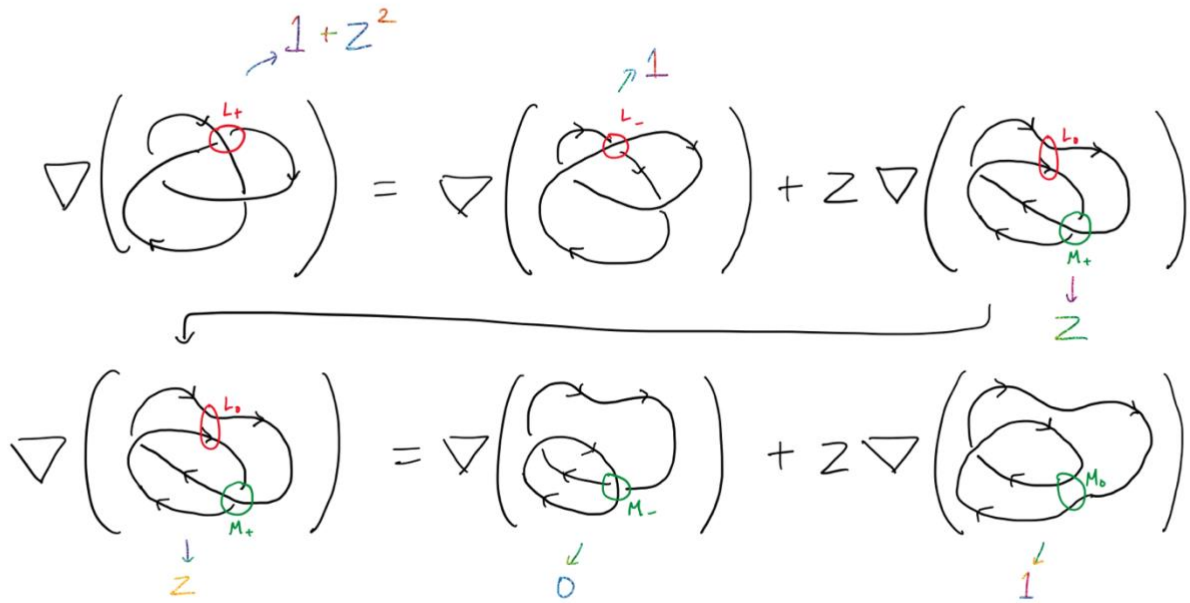
Using the skein relation, you finally come up with the equation:

$$\nabla(L_+) = \nabla(L_-) + z\nabla(L_0)$$

With this relationship, all that changes between whether you start with a left-handed knot or a right-handed knot is the sign of the $z\nabla(L_0)$ term, which is advantageous for your purposes.

After applying this to the trefoil diagram above, you find the resulting polynomial to be $1 + z^2$:



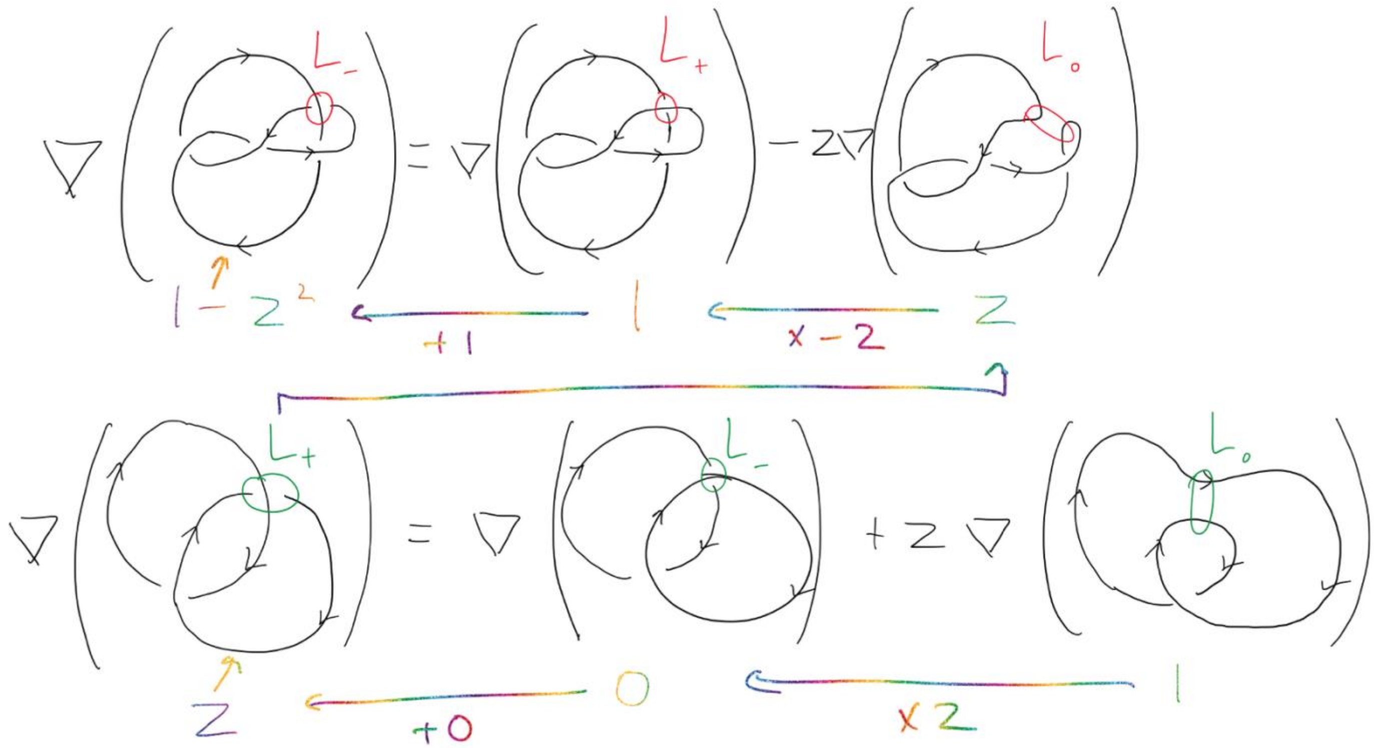


This shows the recursive process of building up a polynomial from the skein relation.

1. *We decompose the trefoil into two different knots, one of which turns out to be the unknot and the other which turns out to be what is known as a Hopf link. We do not yet know the value of this knot, and so we pick a different crossing and decompose it further.*
2. *After doing this process, we end up with two knots that we have defined values for the unlink (value 0) and the unknot (value 1). The unknot is the knot on the bottom right, and is being multiplied by z, and the unlink is the bottom middle knot and has value 0, so we conclude that this particular Hopf link has a polynomial of z.*

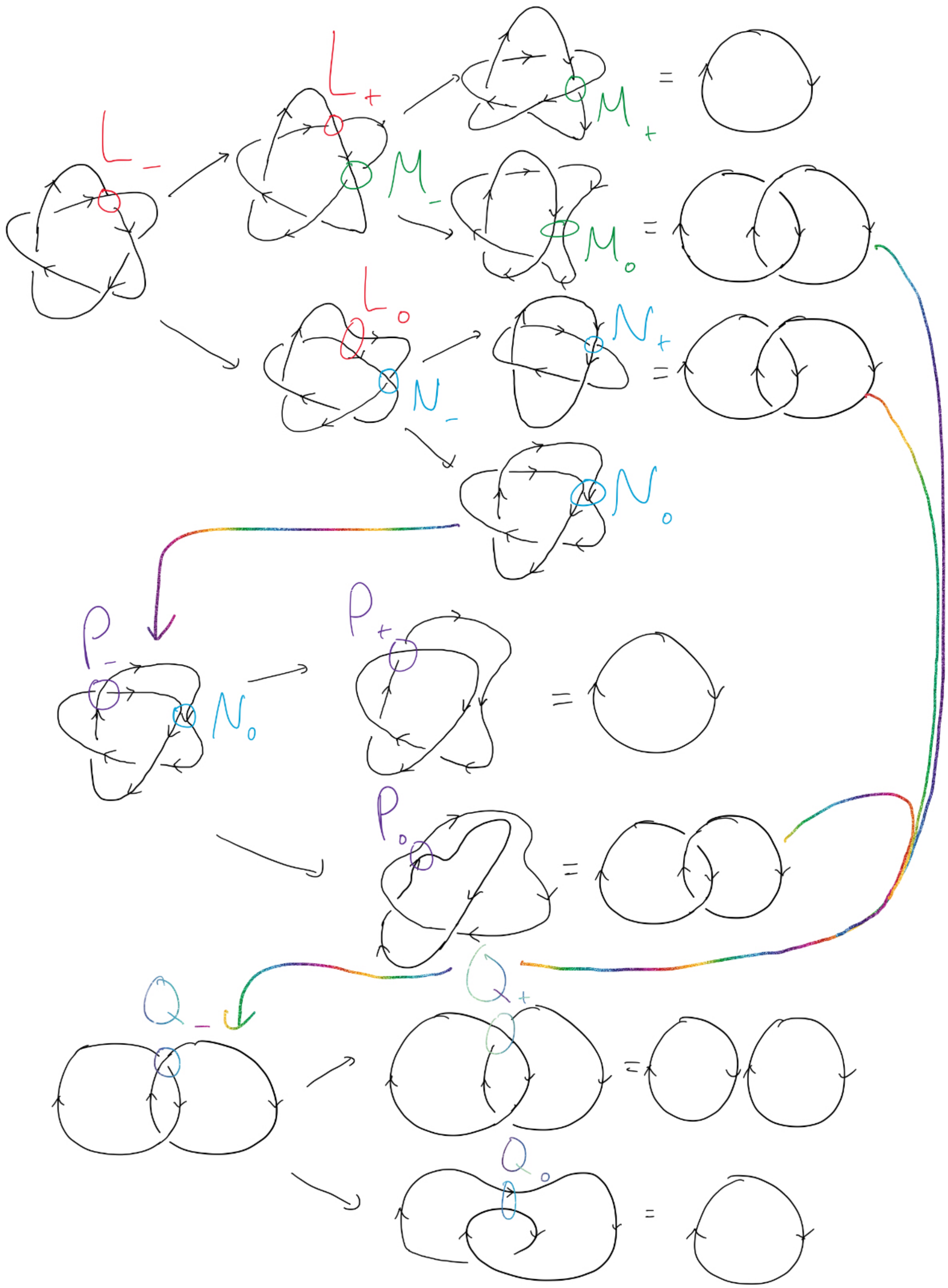
3. *Now that we know the polynomial associated with the Hopf link, we can use that in our first equation: The Hopf link is being multiplied by z, and the other knot in the equation is the unknot, so the final result is that the trefoil has associated polynomial of* $1 + z^2$.

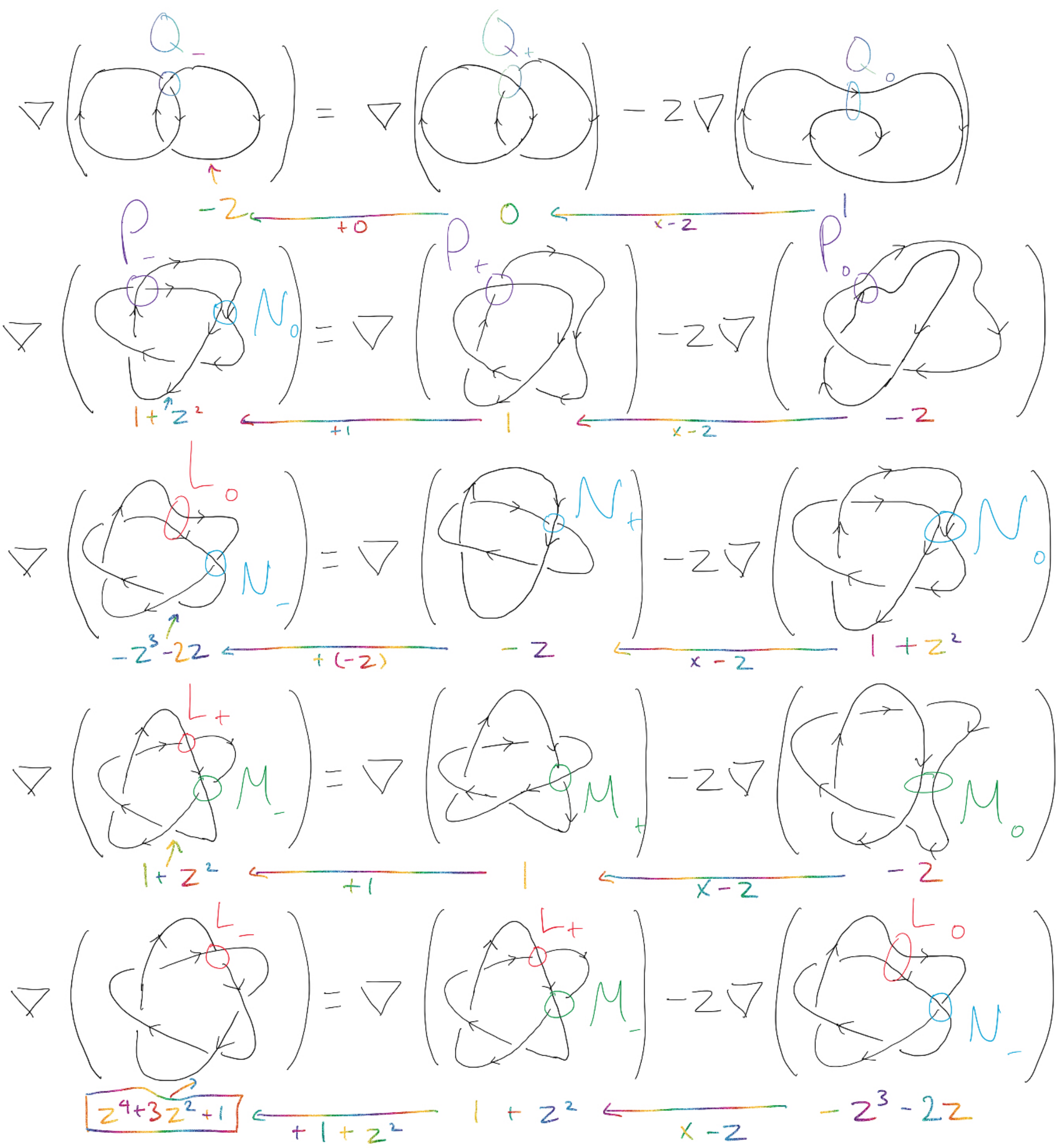
You decide to find the related polynomial (which is known as the Conway polynomial) of the two knots above, hoping that they will be different and that you will have successfully proved that they are fundamentally different knots.



Arrows have been drawn onto the diagram to help highlight the recursive algorithm used to build the polynomial.

And so you have used a method that involves drawing lots of dainty pictures to figure out that the Conway polynomial of the figure 8 knot must be $1 - z^2$. Next, you decide to tackle the cinquefoil, which you dismiss as insignificant and only marginally more difficult, seeing as there is only one single extra crossing to worry about. However, much to your dismay, it was indeed a little more than marginally more difficult.


L_-
L_+
M_+
M_-
M_0
L_0
N_+
N_-
N_0
P_-
N_0
P_+
P_0
Q_-
Q_+
Q_0



Q_-
Q_+
Q_0
-z
+0
0
x -z
1
P_-
P_+
P_0
N_0
1+z^2
+1
1
x -z
-z
L_0
N_+
N_-
N_0
-z^3-2z
+(-z)
-z
x -z
1+z^2
L_+
M_-
M_+
M_0
1+z^2
+1
1
x -z
-z
L_-
L_+
M_-
L_0
N_-
z^4+3z^2+1
+1+z^2
1+z^2
x -z
-z^3-2z

You conclude thus: Since the cinquefoil knot and the figure 8 knot have different Conway polynomials, it follows that they must be fundamentally different knots.

An aside on the Conway polynomial

The Conway polynomial is actually another way to compute the Alexander polynomial, a polynomial invariant obtained when, instead of labelling the arcs with $x_1, x_2, x_3, \ldots, x_n$, you instead label each crossing with $1-t$ for the over-crossing, and t and -1 depending on the right-handedness or left-handedness of the knot, and then follow the normal steps with filling up the entries of a matrix and computing the determinant. This invariant is linked to the Conway polynomial by means of the substitution $\nabla_k\left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right) = \Delta_k(t)$, where $\Delta_k(t)$ is the Alexander polynomial of the knot. The Alexander polynomial is also linked to the colourability of the knot, as $\Delta_k(-1)$ yields the original relationship in modular arithmetic used to compute the determinant of the knot: $t-1$ for the over-crossing gives 2, t gives -1 and -1 just stays as -1. Although the Alexander polynomial can be used to distinguish colourability, it's a stronger invariant than just simple colourability as it can distinguish between more knots. However it fails to tell the difference between knots and their mirror images, which is one of its downfalls. Aside over.

You hold up the sheets of paper filled with messy diagrams to face the screen. After a small delay, the voice erupts from the dual speakers:

"Yes! Your proof is indeed correct! Congratulations, you have just won ten million dollars!"

TASK III COMPLETE

The screen lights up in congratulation, and a fanfare begins playing. The door clicks and swings open with a hiss, and you walk out of the door gleefully.You are glad that you chose to take part in this game show, realising that you have gained the equivalent of ten Millennium Problems in money and in only a fraction of the effort.

Bibliography

Math at Andrews University – Knot Theory

https://youtube.com/playlist?list=PLOROtRhtegr4c1H1JaWN1f6J_q1HdWZOY&si=eF3RHkPS-ZZxL6Yx

Undergraduate Mathematics – Knot Theory

Knot Atlas

http://katlas.org