An Overview of Chaos: Origins, Depictions and Uses

Chaos, according to the dictionary, is a state of complete disorder and confusion. So, how is it we can study something that defies rules and logic, the basis of all mathematics? Something that sounds like it should be completely random?

Because it's not. Chaotic systems are deceptive. They seem random and they look random, but they are not actually random. Chaos theory is the study of just that: seemingly random results and permutations, that are not as random as they appear.

The idea of chaos first came about in the 1890s, when Henry Poincaré was working towards a prize offered by King Oscar II of Sweden and Norway in celebration of his 60th birthday. Poincaré was investigating the motion of planets in three body systems. Newton had already devised an equation for two bodies, and the hope was someone could find a similar one for three.

The consensus at the time was that, in a stable solar system, the planets all followed roughly constant paths, only deviating slightly due to minor disturbances from the other planets. So, according to this model, the future positions of planets ought to be fairly predictable and therefore, calculable.

Poincaré's original entry backed this up, until he realised it contained a mistake. Poincaré had oversimplified some of the calculations and conditions, completely skewing his results. He did own up to this mistake and a different conclusion was published a year later, where he was able to draw some conclusions from what had gone wrong.

Poincaré realised that it was almost impossible to devise a formula for more than 3 bodies that worked every time. A graph showing the permutations of a planets future motion quickly became a visual mess. However, he also realised it was still possible to calculate individual permutations provided you knew the starting conditions.

This became the definition for a chaotic system: pseudo-random, deterministic results. That is, they appeared random overall, yet could be calculated individually.

Following Poincaré's observations, there was little further thought about chaos until the 1960s, when meteorologist Ed Lorenz was running a simulation of atmospheric conditions, using 12 variables to forecast the weather. Lorenz decided he wanted to redo one of the simulations and, to save some time, he decided to cut some corners by simply starting part way through, inputting data from an earlier print off. If he used the same numbers, he would get the same results, right?

This rerun followed the previous simulation for a bit, but then started to diverge. By the end, it was predicting a completely different set of conditions. As it turned out, the computer running the simulation worked to 6 significant figures. But the printed values that Lorenz used were rounded to 3 significant figures. So, such a small, almost negligible, difference in starting values produced wildly different results. This became known as sensitive dependence on initial conditions and, similarly to Poincaré's model of the solar system, gave results that were both deterministic yet practically unpredictable.

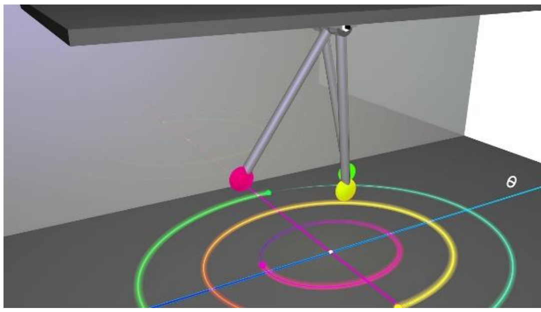
A further consequence of this discovery is that forecasts have a so called "prediction horizon," beyond which models cannot reliably predict conditions. For the solar system, this was tens of millions of years;

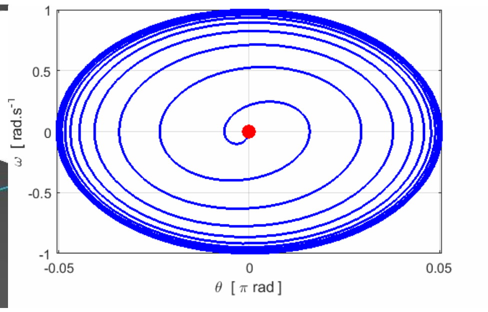
whereas beyond 8 days of simulation, a weather forecast is less accurate than the historical average conditions for that day. Meanwhile, for a double pendulum, this occurs after only a few seconds.

Pendulums are often used to demonstrate chaos. First, you start with a single hinged pendulum, raise it to the horizontal, let go and track the position and velocity of its tip as it slowly returns to equilibrium. A common way of displaying all this, is through a phase space diagram. Phase space is a graph showing a state of being, in this case it is 2D where the x axis represents the angle between the tip and equilibrium while the y axis is its velocity. So, the phase space of a single pendulum is a circle or ellipse, due to simple harmonic motion, that shrinks, eventually converging on the origin. This is fairly intuitive as the pendulum will swing back and forth, with decreasing velocities as kinetic and gravitational potential energy is transferred to the thermal store, meaning the pendulum is slightly lower every time it reaches the end of a swing, until it is finally at rest.

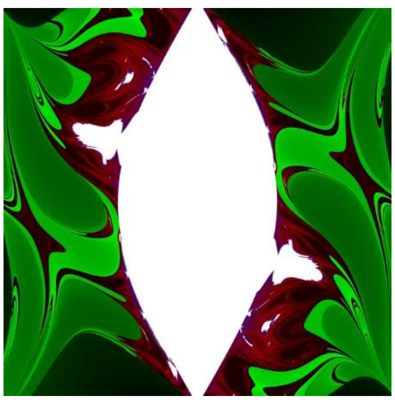
1. (Left) Phase space for different single pendulums – angle from vertical vs velocity

2. (Right) Phase space for a single pendulum – angle from vertical vs angular speed



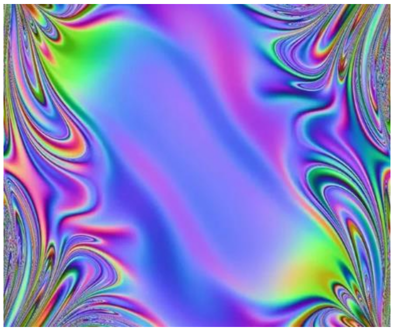


This all changes for a double hinged pendulum. When only displaced slightly, a double pendulum will act roughly the same as a single pendulum. However, when released from greater heights, the double pendulum exhibits chaos. This is due to the second hinge allowing the end part of the pendulum to rotate freely, so the momentum in the end part may keep it moving even when the upper part has slowed or even began to swing back the other way. So, if we tracked the position of the tip, no discernible pattern would emerge. If we plotted a phase space diagram for it though, we would get fractals.



3. Maximum number of oscillations for the pendulum to flip based on initial displacements

For example, by taking the time needed for the double pendulum to flip based on the initial angles of the pendulum, we create this fractal. The x axis is the angle between the upper part of the pendulum and the vertical. Whilst the y axis is the angle between the lower part and the vertical, with the axes intersecting in the middle of the graph. The green colour represents conditions in which either part flips within 10 oscillations, red is within 100 oscillations, purple is within 1000 oscillations and any within 10000 are blue. The large white section in the middle represents those conditions where neither part of the pendulum flips within 10000 oscillations.



4. The development of a double pendulum over time where each combination of angles of displacement are represented by a colour

Another fractal obtained from pendulum swings is made with the same axes as before, but this time assigning every possible combination of angles, and therefore position of the pendulum, a colour. Then, by plotting how every permutation of pendulum develops over time from these initial positions, and updating the colouring with respect to these developments, we get this fractal that grows more complex as time progresses. The colour is only constant for the middle because, as the origin, it represents equilibrium.

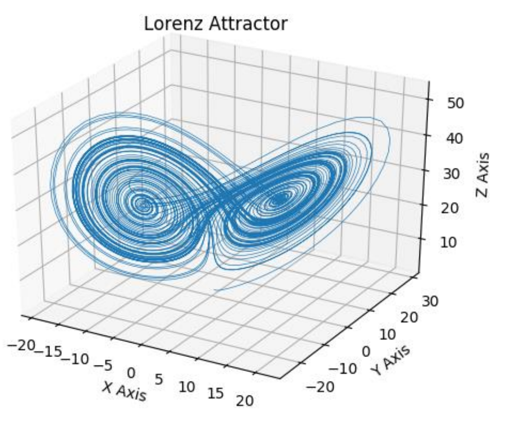
The plotting of chaotic systems leads us onto the idea of attractors. These are images or shapes that the systems create when given enough simulation time. The most famous of these attractors is the 3-dimensional Lorenz attractor, made from the following system of differential equations:

$\frac{dx}{dt}$=σ(y-x)

$\frac{dy}{dt}$=x(ρ -z)-y

$\frac{dz}{dt}$=xy-zβ

In these equations, simplified versions of those used by Lorenz in his weather simulations, σ ,ρ ,β are constants that represent the initial conditions to be inputted. Meanwhile, x represents the rate of convection; y is horizontal temperature variation; and z is the vertical temperature variation.



5. Rate of convection vs Horizontal temperature variation vs Vertical temperature variation under Lorenz's differential equations

The butterfly shaped Lorenz attractor came about when Lorenz used the starting conditions ρ =28,σ =10 and β =8/3. Notably, the path traced out by this attractor never intersect nor do they form a closed loop. Therefore, the path is an infinite curve within a finite space, only possible due to the fractal nature of the chaotic system. This means when calculated to sufficient accuracy, the set of coordinates of each point on the curve is unique.

Interestingly, despite the attractor bearing his name, Lorenz did not coin the term "butterfly effect," which refers to how even the smallest change in initial conditions significantly impacts results.

Further research into the way chaotic systems diverge was carried out by American mathematician, Stephen Smale. He produced the idea of a "horseshoe map" to represent how different data points get so mixed up in chaos. Smale likened data points to banded colours of clay, which he put together and bent into a horseshoe. Then, by flattening and stretching the clay out to its original length, Smale demonstrated that colours that parts of clay that started close together ended up far apart, and vice versa. When repeated several times, the proximity of colours tended to vary dramatically, just as data does when put through a chaotic system.

This leads us to believe that the causes behind chaos are more geometric than algebraic. When the equations are processed, data is stretched such that some points move far away whilst others end up close together, causing fluctuations in results.

One way in which these fluctuations of chaos appear that is perhaps unexpected is in animal and plant populations. According to an article written in 1975 by Australian ecologist Robert May, some equations modelling populations could be chaotic, and major fluctuations in population sizes were not always down to external factors like human interference or climate change. This contradicted the accepted steady state theory of populations, that they grew to a certain size and then just stayed constant if left alone.

Another study in 1995 into the flour beetle, headed by American mathematician James Cushing, found further evidence of chaotic population fluctuation; and a study in 1999 by Dutch biologists Jef Huisman and Franz Weissing on the "Paradox of the Plankton" showed the populations of plankton were also chaotic. In this case, the paradox they were investigating was how there was a greater diversity of plankton species than there was availability of resources. They discovered that this was possible due to the chaos of the different populations: as one population declined, another grew to take advantage of the extra resources. This was repeated often enough that none of the different populations were small enough for long enough to go extinct, disproving definitively the steady state theory of populations, and suggesting that chaos is nature's way of "mixing everything up" - just like in the horseshoe map.

So, it is indisputable that chaos is a major factor in how our world functions, from predicting planetary motion and weather forecasting; to thinking about populations of plankton, chaotic systems are fundamental to how nature operates, and mathematicians have been quick to embrace this.

No longer do weather forecasts rely on just a single simulation, instead they run a myriad of simulations, altering ever so slightly the starting conditions to see the spread of potential outcomes. If the results generally agree, then the weather can be considered predictable, however if they vary from hurricanes and flooding to heatwaves and droughts, then it is less predictable.

Fortunately, rather than just being an inconvenience, the butterfly effect can be utilised in its own way. "Chaotic control" uses the butterfly effect to counter chaos by inputting minor changes into a system, stabilising it and keeping it more predictable. This has a multitude of crucial applications, for example in making designs of heart pacemakers more efficient and less intrusive.

So, in conclusion, despite its confusing appearance, chaos can and has been studied to great length, revealing amazing insights into the workings of our world and patterns that can be appreciated even by those without the mathematical understanding of where exactly they come from. Moreover, these insights are implemented to great effect across a spectrum of professions, from more reliable weather forecasting, to improving medical technology and equipment. Chaos, despite emerging so recently, now undoubtably plays a significant role in all our lives, whether we are aware of it or not, and is a prime example of how advancements in mathematical theory can lead to all manners of advancements elsewhere. And so, this begs the question, what is next? What will the next big revelation be? What great breakthroughs are waiting for some enterprising mathematician to discover them? It may be unpredictable for us, but time will surely tell.

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