Tom Rocks Maths: Essay Competition 2024 Entry
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What is the Optimal Way to Place Oranges on a Grocery Stand? From Circle Packing to Kepler's Conjecture

Imagine yourself in the grocery store, buying oranges. Have you ever wondered how the grocery store places the oranges on the stand? Is it just random or is there a mathematics that lies behind it? Which of the three stacking looks more efficient?

Joel Gordon. "Pyramid (2) Oranges - GOR-65232 -09." *Joel Gordon Photography,* 2009, https://www.joelgordon.com/image/I0000XmYECZzl9r0. Accessed 20 Mar. 2024.

"File:Oranges in Grocery Store." *Wikimedia Commons*, 2017, https://commons.wikimedia.org/wiki/File:Oranges_in_grocery_store_20171221.jpg. Accessed 20 Mar. 2024.

"Orange Fruit Mascot Costume." *Ace Mascot,* 2019, https://www.acemascot.com/orange-fruit-mascot-costume-4604. Accessed 20 Mar. 2024.

It might be easy to say that the third photo does not represent the optimal configuration of oranges on a stand. However, believe it or not, whether the first or the second photograph places the oranges more efficiently has been a topic of mathematical discussion since the 1600s and it took 121 pages to prove Kepler's opinion on the issue.

This essay will cover the concepts of circle and sphere packing to answer the question: What is the best way to place oranges on a grocery stand, and what is the mathematics that lies behind it?

1. **History**

Johannes Kepler was the first person to present his conjecture on optimal sphere packing, i.e., the best way to place spheres within a space such that the spheres are non-overlapping and cover the maximum possible volume within that space. In other words, in a given space, we are looking for the best arrangement of spheres so that they cover most of the volume and do not overlap with each other.

In the early 1600s, "Sir Walter Raleigh, on one of his expeditions, asked a mathematician friend what the most efficient way of stacking the cannonballs he had on the ship was" ("Sphere Packing"). The question was then passed to Kepler, who claimed that the best way to pack the spheres is by placing a layer of spheres as tightly as possible on top of an underlying layer, as shown in Figure 1.1.

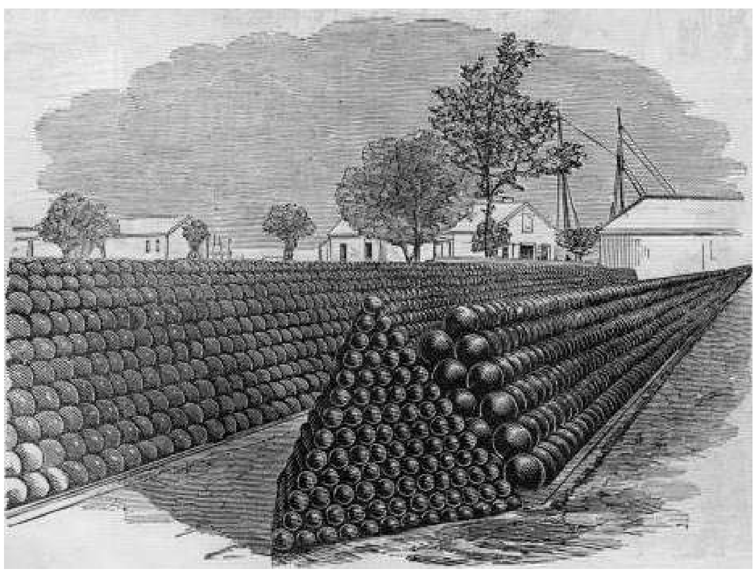

Figure 1.1: "Kepler's Conjecture Proof." *Johannes Kepler,* 2017, https://www.johanneskepler.info/proof-of-keplers-conjecture/. Accessed 20 Mar. 2024.

Although this might seem obvious, the problem was passed on to mathematicians ranging from Gauss to Toth, and finally, a proof of 121 pages was written by Hales. Professors from many prestigious universities, including Princeton and Cambridge, continued their research. While Kepler's conjecture was accepted as the efficient lattice packing (a packing that follows a regular pattern), some cases were told to have optimal packings that are non-lattice. Most recently, in 2022, Maryna Viazovska became the second woman to win the Fields Medal in mathematics, after solving the sphere packing problem for eight dimensions.

The concept of sphere packing has a nearly half-millennium-long history and continues to be one of the most famous mathematical research topics. In this essay, we will discover the basics of it and go over some of the different approaches.

2. Circle Packing in 2D

Before starting with sphere packing in three-dimensional space, let's understand circle packing in two dimensions. The two most famous packing styles are rectangular and hexagonal packing.

Assume you have a 10 cm x 10 cm square grid and circles with diameters of 1 cm each. What is the maximum number of non-overlapping circles you can fit in this square? It is tempting to answer as 100, using rectangular packing; however, the answer is actually 106. How? Let's analyze the different packing methods.

Most of us calculate the number of circles in our heads, using rectangular packing. With this method, the maximum number of circles we can fit on each side is $10\ cm/1\ cm = 10$; thus, the maximum number of circles we can fit on the grid is $10 \cdot 10 = 100$, as demonstrated in Figure 2.1.

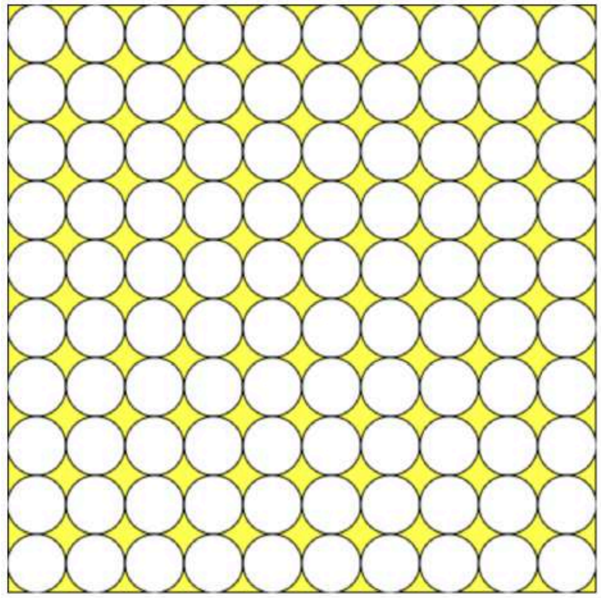

Figure 2.1: Mina Atak. "Rectangular Pattern." *EngineeringToolBox,* 2024, https://www.engineeringtoolbox.com/circles-within-rectangle-d_1905.html. Accessed 20 Mar. 2024.

Now, let's calculate the packing density this arrangement gives.

The area of the grid is $A_{grid} = 10\ cm \cdot 10\ cm = 100\ cm^2$.

The area of one circle is, on the other hand, πr^2, where $r = d/2 = 0.5\ cm$ in this case.

Therefore, the area of one circle is $A_{circle} = \pi(0.5)^2 \approx 0.785\ cm^2$.

This gives the total area the circles cover as $A_{covered} = 100A_{circle} \approx 78.5\ cm^2$. Given the packing density is $\phi = \frac{A_{covered}}{A_{grid}} = \frac{78.5\ cm^2}{100\ cm^2} = 0.785$, 78.5% of the grid's area is covered but $100\% - 78.5\% = 21.5\%$ still remains uncovered.

We can now try the other popular, and in this case, the optimal arrangement for the same grid and circles: hexagonal packing. When placed in the shapes of hexagons, the circles provide the arrangement in Figure 2.2, with 105 as the number of circles placed.

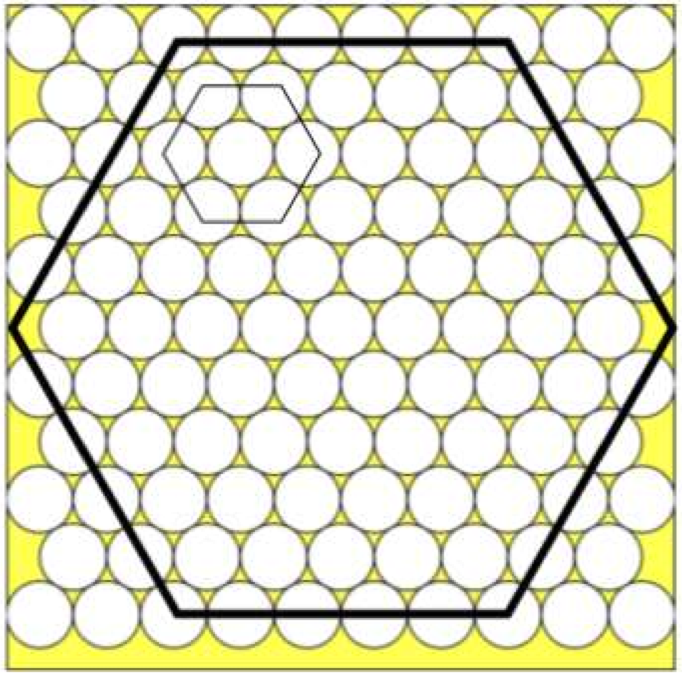

Figure 2.2: Mina Atak. "Hexagonal Pattern." *EngineeringToolBox,* 2024, https://www.engineeringtoolbox.com/circles-within-rectangle-d_1905.html. Accessed 20 Mar. 2024.

This indeed gives a greater number of circles than rectangular packing; however, it seems as if the area left below could also be filled. At this point, the optimal solution is to combine rectangular and hexagonal packing models to try to fit one more circle in that spare area.

The height of a hexagon is equal to $\sqrt{3}$ times its side length, which can be proved through the Pythagorean Theorem (in a right triangle, the sum of the squares of the right sides is equal to the square of the hypotenuse) in Figure 2.3.

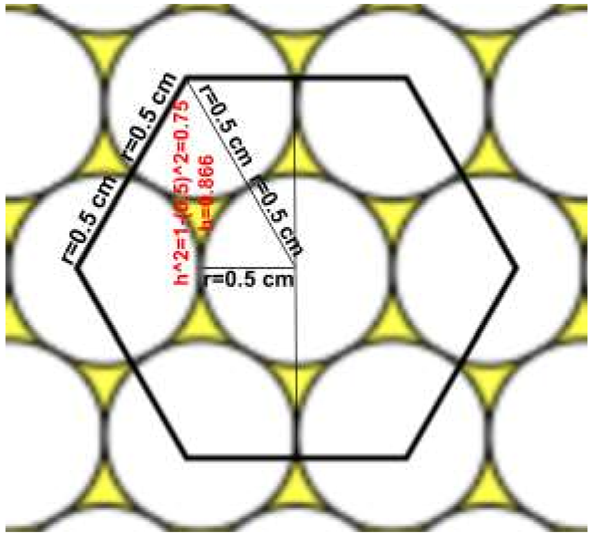

Figure 2.3: Mina Atak. "Hexagonal Pattern with Annotations." *EngineeringToolBox,* 2024, https://www.engineeringtoolbox.com/circles-within-rectangle-d_1905.html. Accessed 20 Mar. 2024.

In our packing model, this corresponds to the height of the hexagon being twice the height of the equilateral triangles inside it, i.e., $2\sqrt{0.75} = 2\frac{\sqrt{3}}{2} = \sqrt{3}$ cm. Hence, in the original hexagonal packing, in terms of the height of the area that is covered, there are five of these $\sqrt{3}$ cm plus the diameter of one circle covered = $5\sqrt{3} + 1 \approx 9.66\ cm$ (Figure 2.4).

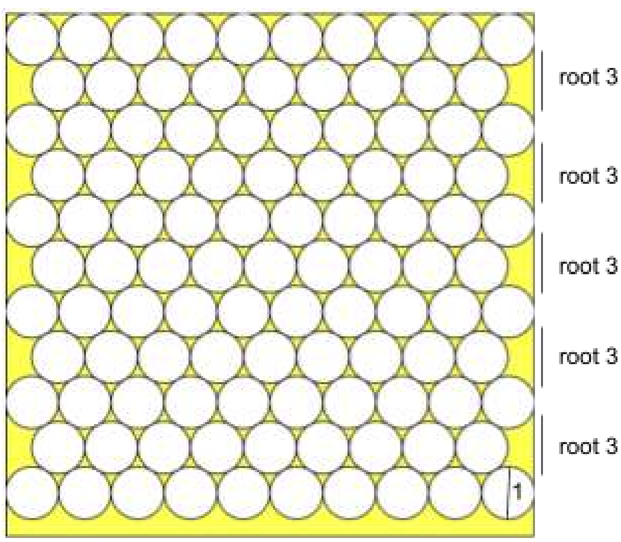

Figure 2.4: Mina Atak. "Hexagonal Pattern's Height." *EngineeringToolBox,* 2024, https://www.engineeringtoolbox.com/circles-within-rectangle-d_1905.html. Accessed 20 Mar. 2024.

If we were to combine this hexagonal packing with rectangular packing, however, and convert the bottom two rows into rectangular placing, we would get the new height covered as $4\sqrt{3} + 3 \approx 9.928\ cm$ (Figure 2.5), still less than but closer to 10 cm!

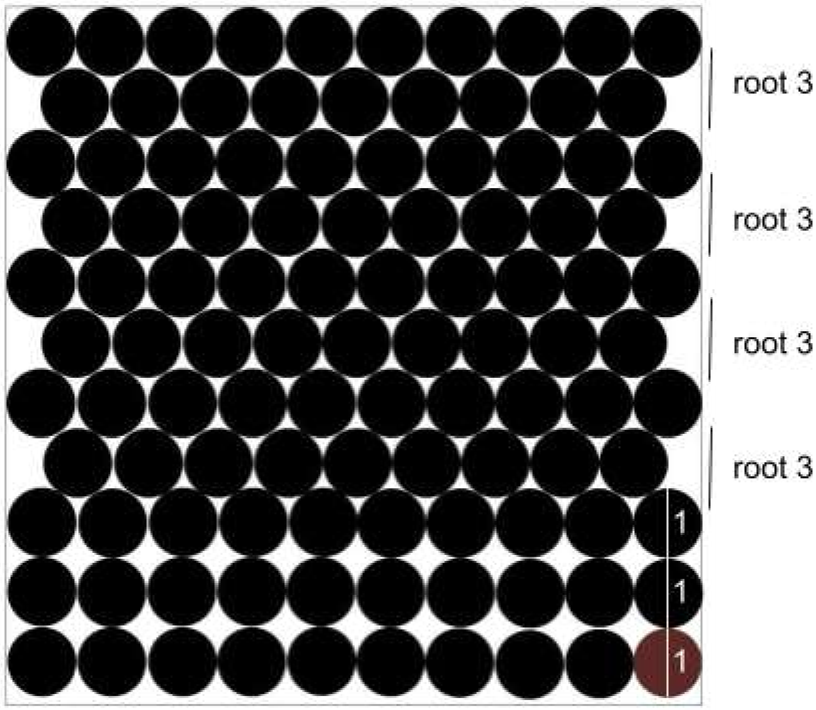

Figure 2.4: Mina Atak. "Ball In A Box with Annotations." *Brilliant,* 2024, https://brilliant.org/problems/ball-in-a-box/. Accessed 20 Mar. 2024.

Converting one more row to rectangular packing wouldn't give us a more optimal arrangement as this would result in the height covered $H = 3\sqrt{3} + 4 \approx 9.196\ cm$, less than what we have just found. This illustrates that the maximum number of circles with d = 1 cm for the 10 cm x 10 cm grid is 106, with a combination of rectangular and hexagonal packing.

Bearing in mind that the optimal configuration will differ for different circles and grids, as we now gained an understanding of basic circle packing, let's move on to our question about oranges and 3D sphere packing! Are you excited?

3. Sphere Packing in 3D

Very similar to circle packing, sphere packing in 3D has methods such as simple cubic packing, face-centered cubic packing, and hexagonal packing.

In simple cubic packing (which is just the 3D version of rectangular packing), one side of the cube is $3d = 6r$, where d is the diameter and r is the radius of one sphere, as portrayed in Figure 3.1.

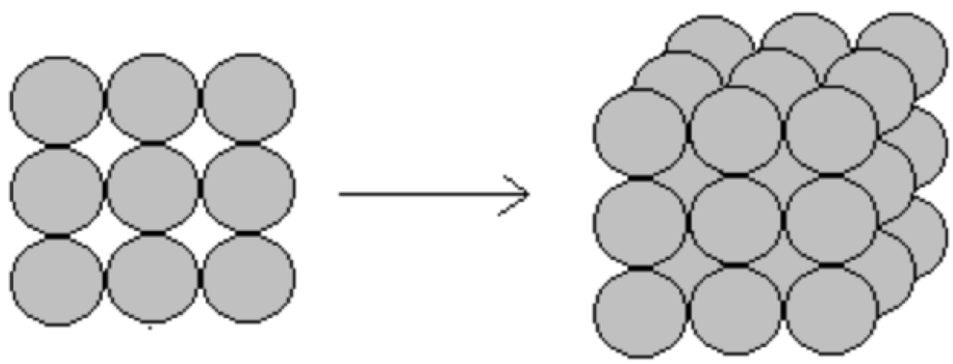

Figure 3.1: M. Beals et al. "Simple Cubic Packing." *Nimbios,* 2000, https://legacy.nimbios.org//~gross/bioed/webmodules/spherepacking.htm. Accessed 20 Mar. 2024.

This makes the volume of the cube $V_{cube} = (6r)^3 = 216r^3$. The volume of one sphere, on the other hand, is $V_{sphere} = \frac{4}{3}\pi r^3$. This gives the total area the spheres cover as $V_{covered} = 27V_{sphere} \approx 36\pi r^3$. Given the packing density is $\phi = \frac{V_{covered}}{V_{cube}} = \frac{36\pi r^3}{216\, r^3} = 0.524$, 52.4% of the cube's area is covered but $100\% - 52.4\% = 47.6\%$ still remains uncovered.

This doesn't sound like the optimal arrangement. Now, let's consider face-centered cubic packing. Face-centered cubic packing is as follows: two equilateral triangles made of six spheres, each with an additional sphere on top are merged back to back, and placed into a cube (Figure

3.2). Attention please, because this is what Kepler claimed in his conjecture to be the most efficient sphere packing method in 3D, with the greatest packing density!

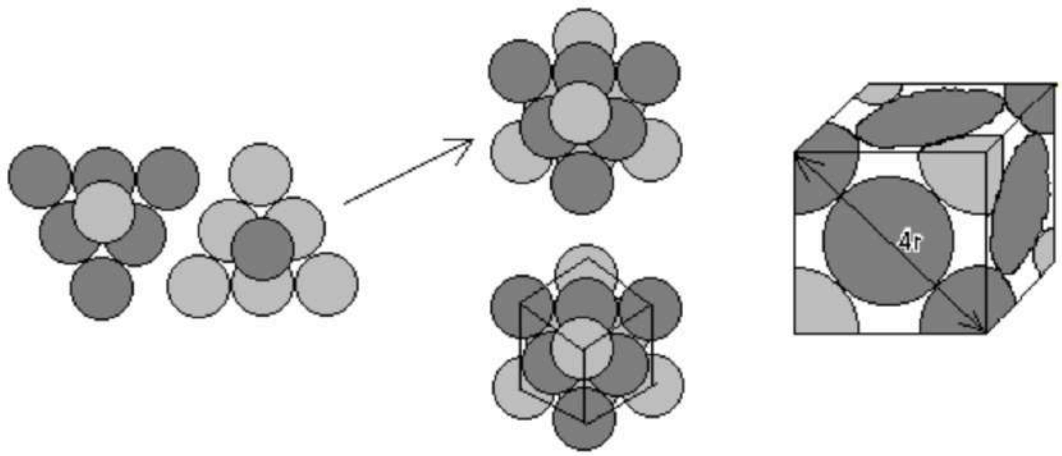

Figure 3.2: M. Beals et al. "Face-Centered Cubic Packing." *Nimbios,* 2000, https://legacy.nimbios .org//~gross/bioed/webmodules/spherepacking.htm. Accessed 20 Mar. 2024.

As seen on the cube at right, this packing will result in "eight 1/8 spheres [or 4 1/4 spheres] in each corner, and six 1/2 spheres at each face" (Beals et al). This makes the total volume covered by the spheres:

$$V_{covered} = 4 \cdot \frac{1}{4} \cdot \frac{4}{3}\pi r^3 + 6 \cdot \frac{1}{2} \cdot \frac{4}{3}\pi r^3 = (1 + 3)\frac{4}{3}\pi r^3 = \frac{16}{3}\pi r^3.$$

The volume of the cube, on the other hand, can be calculated as the cube of its one-side length. In this case, the cube's one-side length can be obtained through the Pythagorean Theorem. As shown in Figures 3.2 and 3.3, the diagonal of the cube's face has a length of $4r$, which gives the cube's side length as:

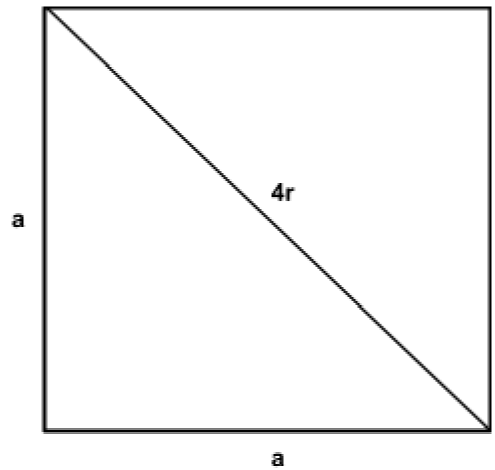

Figure 3.3: Mina Atak. “Cube Face with Annotations.” 20 Mar. 2024.

$$a^2 + a^2 = (4r)^2$$

$$2a^2 = 16r^2\,;\ a^2 = 8r^2$$

$$a = \sqrt{8r^2} = 2r\sqrt{2},$$

Resulting in a cube with volume:

$$V_{cube} = (2r\sqrt{2})^3 = 16\sqrt{2}r^3.$$

Hence, the packing density becomes:

$$\phi = \frac{V_{covered}}{V_{cube}} = \frac{\frac{16}{3}\pi r^3}{16\sqrt{2}\,r^3} = \frac{16}{3}\pi r^3 \cdot \frac{1}{16\sqrt{2}\,r^3} = \frac{\pi}{3\sqrt{2}} = 0.7405,$$

making 74.05% of the cube’s total volume covered and $100\% - 74.05\% = 25.95\%$ uncovered.

This is indeed a higher packing density than that of simple cubic packing and in fact, turns out to be the most efficient method of 3D sphere packing. The third common method, hexagonal packing, has a similar packing density but is difficult to prove through basic geometry. The value we have just calculated, $\frac{\pi}{3\sqrt{2}} = 0.7405$ is what Kepler indicates in his conjecture:

74.05% is the highest possible packing density for congruent, non-overlapping spheres in Euclidean three-space, and "no packing of congruent balls in Euclidean three-space can have density exceeding that of the face-centered cubic packing," i.e, 74.05% (Hales et al).

4. Conclusion

Throughout this essay, we developed our understanding of circle and sphere packing and arrived at the optimal packing density Kepler proposes in his 1611 Conjuncture. This means that we now have the basic knowledge to answer the question we asked ourselves at the beginning of this paper: What is the best way to place oranges on a grocery stand, and what is the mathematics that lies behind it?

Joel Gordon. "Pyramid (2) Oranges - GOR-65232 -09." *Joel Gordon Photography,* 2009, https://www.joelgordon.com/image/I0000XmYECZzl9r0. Accessed 20 Mar. 2024.

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"Orange Fruit Mascot Costume." *Ace Mascot,* 2019, https://www.acemascot.com/orange-fruit-mascot-costume-4604. Accessed 20 Mar. 2024.

Kepler, in fact, answered this question as "face-centered cubic packing," the one represented in the photograph to the furthest left. Mathematically, we can conclude that by placing the oranges as such, the grocery store will be able to cover $\frac{\pi}{3\sqrt{2}} \cdot 100\% = 74.05\%$ of the area it has, a percentage large enough for oranges. Nevertheless, I still think that the third photograph looks fine.

I hope this essay demonstrated that even basic daily life structures have mathematics that lie behind them. As Pythagoras, a famous mathematician, underlined, “There is music in the spacing of the spheres.” Let’s listen to the spheres’ music and follow the path mathematics leads us to!

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