# Cubics to Quantum Waves: The Evolution of Imaginary Numbers

Once upon a time in the realm of mathematics, there existed a problem so perplexing that it confounded the brightest minds for over a millennium. This is not just a story of numbers and equations but a tale of human curiosity, ambition, and the relentless pursuit of knowledge. Our journey begins in the hallowed halls of history, where mathematics was not merely a subject but a language that described the universe. At the heart of this narrative lies an enigma that dared to challenge the very fabric of mathematical reality: Imaginary numbers.

## The Quest Begins: The Cubic Equations

In 1494, amidst the Renaissance's intellectual fervour, Luca Pacioli—Leonardo Da Vinci's esteemed mathematics tutor—published 'Summa de Arithmetica.' This was a compendium of all known mathematics in Italy, featuring a chapter that contemplated the complexities of cubic equations. For 4,000 years, from the ancient civilizations of India, Babylon, Greece, China, and Egypt, mathematicians had been vexed by cubic equations, unable to find a solution. Pacioli himself declared such solutions impossible, a statement that cast a long shadow over the mathematical community.

This declaration was particularly startling considering the progress that had been made with quadratic equations which are formatted as:

$$ax^2 + bx + c = 0$$

These had been solved by mathematicians worldwide, despite their initial resistance to the notion of negative numbers. Their reluctance stemmed from the geometric roots of mathematics; a square with a side length of -3 was inconceivable. However, the quadratic equation was understood, visualised not through numbers but through the dimensions of a square; a negative number was never considered. Hence all the quadratic equations had coefficients which were always positive for example:

$$ax^2 = bx$$

In this equation, there was no possibility of having a $-\ bx$ negative answer; that was just not accepted. This was also because maths at that time was mostly based on geometry and shapes rather than on equations and numbers. So for a quadratic equation, the mathematicians would be visualising a square rather than numbers. So to say that a square has a side of -3 would be impossible and so a negative number was never considered.

Continuing with the journey through mathematical history, the reluctance to embrace negative numbers extended to the realm of cubic equations. Here, Omar Khayyam, a mathematician from Persia, made significant progress by identifying 19 different cubic equations, all characterised by their positive coefficients. Khayyam explored these equations using geometric methods, employing shapes and their intersections to solve some of them. Despite his innovative approach, he recognized the need for further collaboration to fully unlock the secrets of these complex equations.

## A Breakthrough: Scipione Del Ferro's Secret

Four centuries after Omar Khayyam's foray into cubic equations, a breakthrough emerged. Scipione Del Ferro, a mathematics professor at the University of Bologna, discovered a method to solve what are known as depressed cubic equations, which notably lack an $x^2$ term.

Yet, Del Ferro chose a path of secrecy, opting not to share his revolutionary solution with the world. At first glance, this decision might seem perplexing. Why would someone conceal a discovery of such magnitude? The context of the time provides clarity. In Del Ferro's era, the life of a mathematics professor was fraught with competition. Academic positions were often contested in public challenges, reminiscent of duels, where mathematical prowess rather than physical strength determined the victor. In these intellectual battles, mathematicians would pose problems to each other, and the one who solved the most would secure their position, while the other faced professional embarrassment. By keeping his solution secret, Del Ferro protected his career, ensuring he remained indispensable in a highly competitive environment.

Nearly two decades after Del Ferro's discovery, he shared his secret solution with Antonio Fior, a student of his. Fior, while not as adept at mathematics as his mentor, was ambitious. Seizing an opportunity, he challenged Niccolo Fontana Tartaglia, a reputed mathematician who had recently settled in Venice. Understanding the high stakes, Tartaglia and Fior exchanged thirty questions each, all of Fior's being depressed cubics. They were given forty days for the challenge. Fior failed to solve any of Tartaglia's problems, but Tartaglia succeeded in solving all thirty of Fior's. In his quest for solutions, Tartaglia developed his method to tackle the depressed cubics, employing a three-dimensional approach to geometry and becoming the second person to solve these equations. To expedite the process, he even composed the solution algorithm as a poem.

## Unveiling the Secret: Cardano’s Journey from Obscurity to Mathematical Enlightenment

This achievement of Tartaglia's was surely monumental, yet, like Del Ferro before him, he chose to keep the method a secret. This intrigued Gerolamo Cardano, a Milan-based polymath, who persistently sought Tartaglia's solution. Despite Tartaglia's initial refusals and not disclosing any details from his mathematical duel with Fior, Cardano eventually persuaded him to share the secret during a visit to Milan, under the solemn oath that Cardano would never reveal it to others. This agreement marked a pivotal moment in the history of mathematics, as it brought Cardano into the fold of those privy to solving one of the era's most challenging problems.

Armed with Tartaglia's secretive knowledge, Cardano began to toy with the underlying principles, seeking to generalise the solution to encompass all forms of cubic equations, including those with the problematic $x^2$ term. It was during this exploration that Cardano stumbled upon an elegant realisation. By substituting all x terms with $x - \frac{b}{3a}$, he found that the $x^2$ terms were miraculously eliminated:

$$a(x - \frac{b}{3a})^3 + b(x - \frac{b}{3a})^2 + c(x - \frac{b}{3a}) + d = 0$$

$$ax^3 + (c - \frac{b^2}{3a}) + (d + \frac{2b^3}{27a^2} - \frac{bc}{3a}) = 0$$

This simplification, through a stroke of genius, transformed the general cubic equation into a depressed cubic, allowing Tartaglia's formula to be applied universally. Elated by his discovery, Cardano was eager to publish and share this revelation with the world, to stand among the giants in mathematics. His position as a physician and intellectual meant that he was not bound by the same constraints that plagued career mathematicians; for Cardano, the pursuit of recognition was more compelling than clinging to secrecy.

## Cardano’s Dilemma: The Discovery of Imaginary Numbers

However, Cardano faced a moral dilemma. He was bound by an oath of silence to Tartaglia, an oath that was now in direct conflict with his desire to illuminate the mathematical community with his findings. Seeking a resolution that would keep his conscience clear, Cardano journeyed to Bologna. There, he met with the son-in-law of Scipione Del Ferro, who had originally solved the depressed cubics. In the archives of Del Ferro's notes, Cardano found independent confirmation of the solution, freeing him from his promise to Tartaglia and setting the stage for one of the most influential mathematical publications of the time: Ars Magna. In this treatise, Cardano dedicated a chapter to each potential form a cubic equation could take, acknowledging not only his findings but also the contributions of Del Ferro, Fior, and Tartaglia.

Amidst this scholarly endeavour, Cardano encountered certain cubic equations that seemed to defy solution, presenting no real answers. An example of such a stubborn equation was:

$$x^3 = 15x + 4$$

Cardano, applying his own method, found himself facing a radical containing a negative number:

$$\sqrt[3]{2+\sqrt{-121}}+\sqrt[3]{2-\sqrt{-121}}$$

Confronted with this conundrum, Cardano reached out to Tartaglia, who dodged the issue, suggesting that Cardano had misapplied the formula. The truth was that even Tartaglia, who had evaded revealing his methodology, could not have foreseen this complication. Cardano, seeking answers, revisited the geometric origins of the solution and realised the necessity to incorporate negative areas, a concept that lay beyond the grasp of geometry at the time. In doing so, he stumbled upon the root of negative numbers—an area of mathematics that had not yet been charted.

This peculiar case of the cubic equation was baffling. Despite the nonsensical nature of the negative radicand according to the knowledge of the time, trial and error revealed a real solution: $x$=4. Cardano's conundrum, wherein his method that succeeded with other cubic equations failed for certain cases, marked a pivotal moment in mathematical history. He ultimately chose to sidestep the problematic equation in his "Ars Magna," leaving the mystery unsolved in the published work.

It was approximately ten years later that the Italian mathematician Raphael Bombelli bravely stepped beyond Cardano's final point of inquiry. Bombelli decided to confront the square roots of negatives head-on, not as anomalies to be avoided, but as entities to be explored and understood.

Bombelli reimagined the equation Cardano had set aside:

$$x^3 = 15x + 4$$

He approached the solution through Cardano's method, arriving at a point where he encountered the radical expression:

$$\sqrt[3]{2+\sqrt{-121}}+\sqrt[3]{2-\sqrt{-121}}$$

Bombelli saw past the negative square root, treating it as a variable in its own right. He posited the equation in a form that recognized these radicals as a legitimate part of the solution:

$$\sqrt[3]{2+\sqrt{-121}} = a + b\sqrt{-1}$$

$$\sqrt[3]{2-\sqrt{-121}} = a - b\sqrt{-1}$$

This allowed him to simplify the expression to:

$$2+\sqrt{-1}+2-\sqrt{-1}$$

Here, the $\sqrt{-1}$ terms cancelled each other out, leaving him with a real number solution for *x*, which was indeed 4, validating Cardano's method.

This revelation demonstrated that the geometric proofs, once the bedrock of algebra, were no longer the only path to mathematical truth. Over the ensuing century, mathematics underwent a revolution. Symbolic notation introduced by François Viète in the 1600s replaced lengthy geometrical and verbal descriptions. Geometry, while still central, was no longer the sole foundation upon which mathematics was built.

But it wasn't until Descartes dismissively labelled these numbers as 'imaginary' that they began to acquire their modern identity. Descartes' designation, meant to signify their lack of physical meaning, instead provided a linguistic foothold that allowed them to be discussed more concretely. And, with Euler's introduction of the symbol *i* to represent $\sqrt{i}$, a bridge was built that allowed mathematicians to cross from the real to the complex plane. The mathematical community was, at last, poised to accept these new numbers not as figments of mathematical aberration but as legitimate entities.

## Embracing Imaginary Numbers

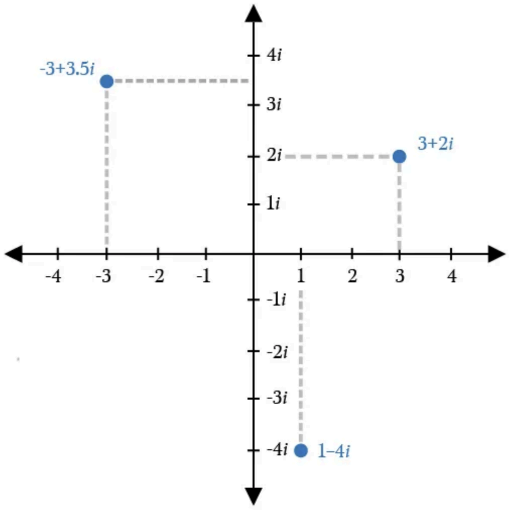
The breakthroughs of the 16th century had been marvellous, but they were merely the prologue to an even grander narrative. As the centuries passed, the concept of the square root of -1, continued to simmer in the minds of the greatest thinkers. Fast forward to 1925: Erwin Schrödinger, a visionary scientist, was on a quest to describe the behaviour of quantum particles. Building on de Broglie's insight that matter possesses wave-like properties, Schrödinger developed one of the most profound equations in physics—the Schrödinger equation:

$$i\hbar \frac{\partial}{\partial t} \psi(\mathbf{r}, t) = -\frac{\hbar^2}{2m} \nabla^2 \psi(\mathbf{r}, t) + V(\mathbf{r}, t)\psi(\mathbf{r}, t)$$

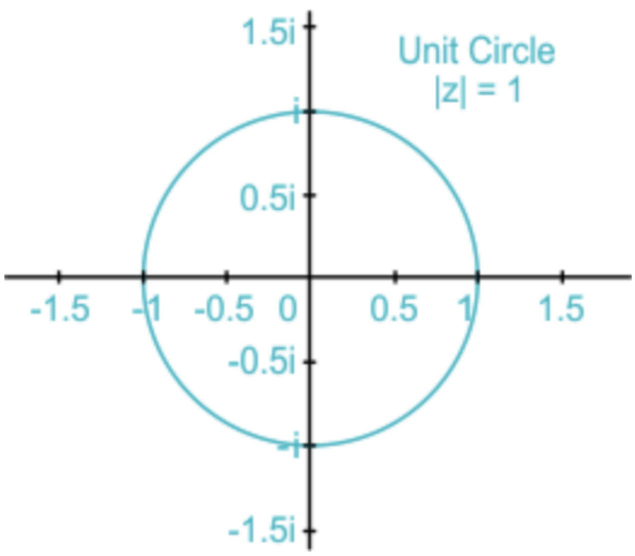
Here, the imaginary unit *i* is not merely a placeholder but a fundamental feature, highlighting the inseparable role of imaginary numbers in the quantum realm. Mathematicians had grown accustomed to these surreal numbers appearing in abstract equations, but for physicists, their emergence in such a foundational theory was startling. Even Schrödinger himself expressed reservations, initially viewing the wave function as fundamentally a real function and questioning the use of complex numbers in his equation.

## The New Number Line: The Complex Plane

To unravel this enigma, one must first visualise the number system without imaginary numbers—considering only the real number line where positives stretch to the right and negatives to the left, housing rational and irrational numbers alike. However, to accommodate imaginary numbers, envision a plane: the Cartesian plane with the real number line on the x-axis and the imaginary numbers, multiples of *i*, on the y-axis.




When we multiply by *i* continuously, starting from the point (0,1) on this plane, we observe a rotational movement: from 1 to *i*, then to -1, to −*i*, and back to 1, completing a full circle around the origin (0,0). This cyclical behaviour is captured elegantly by Euler's formula:



which describes a spiral when $x$ varies over time, linking the realms of exponential growth to rotational motion.

In the real world, this equation's real part traces a cosine wave, but when we consider its full complex form, it describes both the cosine and sine waves, encapsulating the essential wave functions in one elegant expression. This raises a fundamental question: why rely on $e^{ix}$ instead of simply using a sine function? The exponential form offers a valuable property—its derivative with respect to time or position is proportional to the function itself, a trait not shared by sine and cosine functions, whose derivatives cyclically interchange.

The Schrödinger equation, completed with its use of the imaginary unit, encapsulates our understanding of atomic behaviour, underpins the vast structure of chemistry and much of physics. This implies that nature doesn't adhere strictly to the real numbers; instead, it operates within the complex number framework, where the once elusive $\sqrt{-1}$ now plays a pivotal role.

## Conclusion

What started as an abstraction to solve cubic equations has transcended its origins, becoming indispensable in our mathematical description of the very fabric of reality. The quest to solve the cubics became one of unending, infinite discovery; one that evolved from seeking answers to deciphering and redefining the language of the universe as we know it. Perhaps this is what Galileo meant when he poetically observed, 'Mathematics is the language in which God has written the universe.'

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