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Pingala's Poetry Puzzle

Introduction:

Despite its reputation as a tedious and uninspiring subject, I consider mathematics to be one of the most creative subjects there is. While we cannot control what an answer turns out to be, the art is in the path we take to connect one truth to another. This takes lateral thinking and insight, and this insight is most beautiful, in my opinion, when motivated by a real-world problem. My favourite example of this has its roots in ancient Indian literature, in the study of Sanskrit prosody.

Sanskrit Prosody:

Sanskrit is an ancient language, spoken in various forms between roughly 400BCE to 1350CE in South Asia. While there are no native speakers left, it is still very important as the language of the Vedas, the oldest scriptures of Hinduism. Sanskrit prosody, also known as Chandas, is one of the six Vendangas, traditional fields of study surrounding the Vedas. This field studies poetic metre and verse in Sanskrit, particularly that of the Vedas themselves. Traditionally, it is tightly linked to mathematics, due to the importance of structure and patterns.

A verse in Sanskrit poetry is generally split into four lines, known as padas, and each line is constructed of a few morae, known as matras. These morae either come alone, as a short syllable (laghu), or in a pair, as a long syllable (guru). There are three major types of metre, and one type known as matra-vrttas lead to an interesting mathematical puzzle. A line in matra-vrtta consists of a constant total number of morae, but the number and arrangement of syllables is arbitrary. For example, a line with two morae could either have two short syllables or one long syllable. I will represent these lines with the diagrams below, where a white square represents a short syllable, and a green rectangle represents a long syllable, equivalent in width to two short syllables (1).

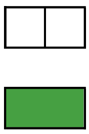

Fig. 1: A line with two morae can have either two short syllables or one long syllable.

One poet, named Pingala, invented a lot of incredibly innovative mathematics to help him study Sanskrit prosody. He is often credited with the invention of the binary number system and being the first to treat 'zero' as a number. In studying matra-vrttas, he wondered how many different ways there are to construct a line with a given number of morae, m. For example, a line of 5 morae can be arranged in 8 ways, as in Figure 2.

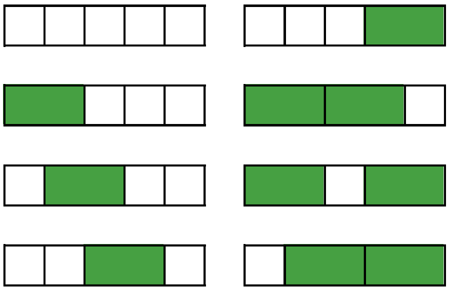

Fig. 2: There are eight ways to construct a line with five morae.

Attacking the Puzzle:

The first method you might take to try to solve this problem is by attempting to systematically write down all the different ways to construct a line. One way to do this is by considering increasing numbers of long syllables. You might start by investigating how many ways there are to include 0 long syllables. The answer will always be one, as every way of reordering short syllables is the same. Next, you could consider the possibilities with exactly 1 long syllable. Here you would find that that there are $m-1$ different placements of long syllables for m morae, as this syllable could be placed in any of the m positions except the last. You could continue by considering 2 long syllables, determining all the places you can put the second long syllable if the first is at the start of the line, by moving the second along each available slot until you reach the end, as shown in Figure 2. Each time the second long syllable gets to the end of the poem, you can then move the first long syllable right by one position, and repeat, until there is no more space. Remember that the second long syllable must not go before the first, as this possibility will already have been covered. This method can be replicated for three or more long syllables, however the complexity increases rapidly as you add more long syllables. This makes it difficult to predict the total number of arrangements you will find, even though this searching method is exhaustive. Clearly, it would be convenient if there were a simpler way to count the possibilities.

Pingala spotted a pattern that connects different numbers of morae. Let us define some sequence U such that U_m returns the number of arrangements of a line of m morae. We can observe that every line must either end in a short or a long syllable, leading us to realise that U_m is equal to the number of arrangements that end in a short syllable plus the number of arrangements that end in a long syllable. This simple idea allows us to find a much easier way to count the number of arrangements, as long as we can count how many arrangements end in each type of syllable. First, considering how many arrangements end in a short syllable, we notice that we can remove the short syllable from the end, and now we have another line, this time with $m-1$ morae. This gives us the same problem, just with a shorter line. We will just write this as U_{m-1} for now. Moving on to the set ending in the long syllable, we can again just take it off, which leaves us with a line of $m-2$ morae, as the long syllable consists of 2 morae. The number of these combinations that exist is therefore U_{m-2}. Since we know that U_m is the sum of the number ending in a short syllable and the number ending in a long syllable, we can write that $U_m = U_{m-1} + U_{m-2}$. Figure 3 shows this method of counting the arrangements of a line with five morae.

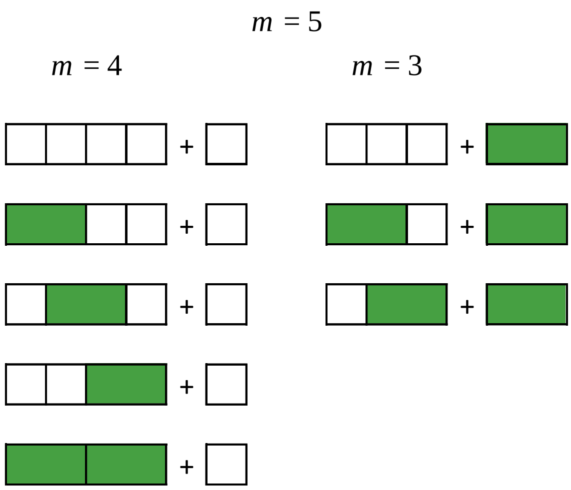

Fig. 3: A different way to find all the 5-long lines, by dividing them up into 4-long lines with an added short syllable and 3-long lines with an added long syllable.

The formula $U_m = U_{m-1} + U_{m-2}$ applies for all positive integer values of m except 1 and 2, since you would have to find U_{-1} and U_0 which do not make sense, as you cannot have a line with negative or zero morae. Therefore, we have to define that $U_1 = 1$ and $U_2 = 2$, and then we can calculate U_m for all $m \geq 3$ using the recursive formula.

m	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
U_m	1	2	3	5	8	13	21	34	55	89	144	233	377	610	987

Fig. 4: The value of U_m can be found much more easily using the recursive formula than by listing every possible arrangement.

As we can see, by 15 morae there are already almost one thousand ways to arrange the line of the poem. It would be much harder to count each of these individually than to use the recursive method. We can simply find U_3 as $U_2 + U_1$, then U_4 as $U_3 + U_2$, and so on until we reach U_{15}, which only takes 13 simple additions.

At this point, you may have felt like this sequence feels vaguely familiar. This sequence is very similar to the famous Fibonacci sequence. The Fibonacci sequence, F, is defined such that $F_n = F_{n-1} + F_{n-2}$ for all values of $n \geq 3$, the same recurrence relationship as in the sequence U. However, the Fibonacci sequence is defined such that $F_1 = 1$ and $F_2 = 1$, whereas the sequence U defines that $U_1 = 1$ and $U_2 = 2$. As $F_3 = F_2 + F_1 = 2$, this means that $U_1 = F_2$ and $U_2 = F_3$. Since we can calculate the next element of each sequence by adding two consecutive previous terms, we can see that the rest of the U sequence will be the same as the F sequence, except that $U_m = F_{m+1}$, so the sequences are one place out of line.

A Better Answer:

While a useful tool to solve our poetry puzzle, the recursive method still feels tedious. It would be better to jump straight to an answer for a given number of morae, without first answering it for 3, 4, 5, and so on. To define precisely what we mean by this mathematically, however, we need to define a few terms first. For the sake of this essay, I will refer to the relationship: $U_m = U_{m-1} + U_{m-2}$ as 'the

Pingala recursion', and any sequence that follows this relationship as a 'Pingala sequence'. The term 'the Fibonacci sequence' will be used to describe the specific Pingala sequence, F, where $F_0 = 0$ and $F_1 = 1$. Notice that this is a slightly different definition of the Fibonacci sequence to the one I previously used, where we defined F_1 and F_2. These alternative definitions lead to exactly the same sequence beyond F_0, however defining F_0 will make the maths slightly easier, and is the modern convention. Therefore, to define what we mean by a 'simpler' way to solve Pingala's puzzle, we can ask: Can we find a formula for F_n where any reference to F has a constant subscript? This limitation means that we can only define F_n in terms of some pre-defined initial values and n itself. The advantage of this is that we can cut straight to the chase, without having to iterate to find our result. An expression like this is called a 'closed-form expression'.

If you have ever tried to find such an expression for the Fibonacci numbers, you will know that it is not as simple as it sounds. It can feel like the only obvious pattern in the numbers is how they relate to the ones next to them, while seeming completely unrelated to their actual position in the sequence. In this case, it is helpful to step back, and study Pingala sequences more generally, to see if this approach can give us any better insights.

Operations on Pingala Sequences:

Let us consider doing different operations on Pingala sequences. A reasonable question is whether one will stay a Pingala sequence after the operation. There are two particularly useful operations, starting with multiplying each term by a constant, k. Let U be an arbitrary Pingala sequence, and V be the sequence such that $V_n = kU_n$. By definition:

$$U_n = U_{n-1} + U_{n-2}$$

We can multiply both sides by k:

$$kU_n = k(U_{n-1} + U_{n-2})$$

$$kU_n = kU_{n-1} + kU_{n-2}$$

As V_n is defined as kU_n we can rewrite this as:

$$V_n = V_{n-1} + V_{n-2}$$

Therefore we have shown that V is also a Pingala sequence as it satisfies the Pingala recursion.

Another operation we could consider is adding together two Pingala sequences. If we have two Pingala sequences, U and V, and another sequence W such that $W_n = U_n + V_n$, we can find an alternative expression for W_n by rewriting U_n and V_n with the Pingala recursion:

$$W_n = (U_{n-1} + U_{n-2}) + (V_{n-1} + V_{n-2})$$

Regrouping the terms:

$$W_n = (U_{n-1} + V_{n-1}) + (U_{n-2} + V_{n-2})$$

We can now use the definition of W, replacing each bracket with a term of W.

$$W_n = W_{n-1} + W_{n-2}$$

Again, we have shown that W obeys the Pingala recursion and is therefore a Pingala sequence.

Now we have two different ways to manipulate Pingala sequences without disobeying the Pingala recursion, and combining these serves as a very powerful tool for generating new Pingala sequences, by defining sequence W in terms of Pingala sequences U and V as:

$$W_n = aU_n + bV_n \tag{1}$$

where a and b are arbitrary constants. We know that W_n must be a Pingala sequence as each term in the addition is due to the multiplication rule, and they add to a Pingala sequence thanks to the addition rule. In fact, this formula is so powerful that just by adjusting the choice of a and b, we can create any possible Pingala sequence.

Constructing Any Pingala Sequence:

If we know the first two terms of a Pingala sequence, U_0 and U_1, we can find the rest of the sequence using the Pingala recursion. I like to say that they are 'deterministic' beyond the second item: if you know all previous items exactly, you can also know all future items exactly. Therefore, we can uniquely define any Pingala sequence by its first two values. Two Pingala sequences are different if and only if the first two values are different. Therefore, if we can manipulate a and b so that W_0 and W_1 are given the correct values, we can be sure that every subsequent term in the sequence will be given the correct value too. This leads us to two simultaneous equations in a and b:

$$\boxed{1}: \qquad W_0 = aU_0 + bV_0$$
$$\boxed{2}: \qquad W_1 = aU_1 + bV_1$$

By multiplying the first equation by $\frac{V_1}{V_0}$ we obtain:

$$\frac{V_1}{V_0}\boxed{1}: \qquad \frac{V_1W_0}{V_0} = \frac{V_1U_0}{V_0}a + V_1b$$

Now subtracting each side by the corresponding side of the second equation we get:

$$\frac{V_1}{V_0}\boxed{1} - \boxed{2}: \qquad \frac{V_1W_0}{V_0} - W_1 = \frac{V_1U_0}{V_0}a - U_1a$$
$$\left(\frac{V_1}{V_0}\boxed{1} - \boxed{2}\right)V_0: \qquad V_1W_0 - W_1V_0 = (V_1U_0 - U_1V_0)a$$

$$a = \frac{V_1W_0 - W_1V_0}{V_1U_0 - U_1V_0} \tag{2}$$

We can follow a similar method to find b:

$$\frac{U_1}{U_0}\boxed{1}: \qquad \frac{U_1W_0}{U_0} = U_1a + \frac{U_1V_0}{U_0}b$$
$$\frac{U_1}{U_0}\boxed{1} - \boxed{2}: \qquad \frac{U_1W_0}{U_0} - W_1 = \frac{U_1V_0}{U_0}b - V_1b$$
$$\left(\frac{U_1}{U_0}\boxed{1} - \boxed{2}\right)U_0: \qquad U_1W_0 - W_1U_0 = (U_1V_0 - V_1U_0)b$$

$$b = \frac{U_1W_0 - W_1U_0}{U_1V_0 - V_1U_0}$$

$$b = \frac{U_0W_1 - W_0U_1}{V_1U_0 - U_1V_0} \tag{3}$$

Therefore, we have found a formula for a and b that can give us any desired Pingala sequence W as long as we already have two Pingala sequences U and V, and we know what we want for W_0 and W_1. This only does not work if $V_1U_0 - U_1V_0 = 0$ as this is the denominator, and we cannot divide by zero. This occurs if $U_n = kV_n$ where k is a constant, as the expression is then written: $V_1kV_0 - kV_1V_0$ which is clearly zero. This cannot work because adding constant multiples of the same sequence will simply result in another constant multiple of the same sequence. Constant multiplication is insufficient to find all Pingala sequences as the ratio of the starting terms of a series is preserved under a constant multiplication, however any ratio could exist between the starting terms in an arbitrary Pingala sequence.

Finding Examples of Pingala Sequences:

We may feel no closer to finding a closed-form formula for the Fibonacci numbers, since this method relies on having a closed-form formula for two Pingala sequences, which seems harder than doing so for one. However, we are in a slightly different situation, as we have removed any starting conditions for sequences U and V, meaning all we have to do is find two sequences that satisfy the Pingala recursion with no concern for their first two values.

One idea is a sequence of only zeroes. While this does satisfy the Pingala recursion, and is therefore a Pingala sequence, unfortunately it is not useful for our purpose. This is because it is a constant multiple, 0, of every other Pingala sequence. Therefore, it is not viable in our general Pingala expression.

Instead, let us invent a function I such that $I(U_n) = U_{n+1}$. This function acts as an 'incrementor' operation on U, meaning that it gives the next item of U given the current one. Using the Pingala recursion:

$$U_n = U_{n-1} + U_{n-2}$$

$$I(U_n) = I(U_{n-1} + U_{n-2})$$

$$U_{n+1} = I(U_{n-1} + U_{n-2})$$

By definition, we also know that $U_{n+1} = U_n + U_{n-1}$, so we can write that:

$$U_n + U_{n-1} = I(U_{n-1} + U_{n-2})$$

The left-hand side is now the same as individually incrementing each term. Therefore:

$$I(U_{n-1}) + I(U_{n-2}) = I(U_{n-1} + U_{n-2})$$

Looking at the structure of this equation, it looks very similar to the process of factorisation. Therefore, a reasonable hypothesis is that I is a multiplication by some constant x. Following this hypothesis:

$$xU_n = U_{n+1}$$

Consequently, we can see that U_n must be found by repeated multiplication by x, starting at U_0, which is simply exponentiation, meaning that a possible expression for U_n could be $U_n = x^n$. This allows us to rewrite our Pingala recursion for this sequence as:

$$x^n = x^{n-1} + x^{n-2}$$

This leaves us with an equation to find our made-up constant x. By dividing through by x^{n-2} we are left with a simple quadratic equation:

$$x^2 = x + 1$$

Using the quadratic formula, we can solve this equation to get:

$$x = \frac{1 \pm \sqrt{5}}{2}$$

This gives us two possible values of x, which also means that we arrive with two different sequences that both satisfy the Pingala recursion, based on which value of x we choose. These values are actually very common numbers in mathematics. The positive one is the 'golden ratio', written φ, and roughly equal to 1.618. The negative one is less well-known, but is also very important. It is denoted ψ and is roughly equal to -0.168. Exactly:

$$\varphi = \frac{1 + \sqrt{5}}{2}$$

$$\psi = \frac{1 - \sqrt{5}}{2}$$

These numbers allow us to define two Pingala sequences, U and V:

$$U_n = \varphi^n$$

$$V_n = \psi^n$$

The General Expression for a Pingala Sequence:

These two sequences let us generate any possible Pingala sequence, using the simultaneous equations from before. We can first evaluate the first two terms for each sequence:

$$U_0 = \varphi^0 = 1 \qquad U_1 = \varphi^1 = \varphi$$
$$V_0 = \psi^0 = 1 \qquad V_1 = \psi^1 = \psi$$

Plugging these into our general Pingala sequence expression, equation (1):

$$W_n = aU_n + bV_n$$
$$W_n = a\varphi^n + b\psi^n$$

And we can solve for a using equation (2):

$$a = \frac{V_1 W_0 - W_1 V_0}{V_1 U_0 - U_1 V_0}$$

$$a = \frac{W_0 \psi - W_1}{\psi - \phi}$$

Using the exact values of φ and ψ we can also write this as:

$$a = \frac{W_1 - W_0 \psi}{\sqrt{5}}$$

Similarly, we can find b, using equation (3):

$$b = \frac{U_0 W_1 - W_0 U_1}{V_1 U_0 - U_1 V_0}$$

$$b = \frac{W_1 - W_0\varphi}{\psi - \varphi}$$

$$b = \frac{W_0\varphi - W_1}{\sqrt{5}}$$

Finally, we can plug these back into the main formula, equation (1):

$$W_n = \left(\frac{W_1 - W_0\psi}{\sqrt{5}}\right)\varphi^n + \left(\frac{W_0\varphi - W_1}{\sqrt{5}}\right)\psi^n$$

$$W_n = \frac{(W_1 - W_0\psi)\varphi^n + (W_0\varphi - W_1)\psi^n}{\sqrt{5}} \qquad (4)$$

Therefore, this formula can be used to find the n^{th} term of any Pingala sequence, provided we know the first two terms, W_0 and W_1. We can plug in F_0 and F_1 in order to find the n^{th} term of the Fibonacci sequence:

$$F_n = \frac{(1 - 0\psi)\varphi^n + (0\varphi - 1)\psi^n}{\sqrt{5}}$$

$$F_n = \frac{\varphi^n + \psi^n}{\sqrt{5}} \qquad (5)$$

Solving the Puzzle:

We can also use this formula to solve Pingala's poetry puzzle for a line with m morae. While this sequence, U, is defined in terms of U_1 and U_2, rather than U_0 and U_1, we can make up a value for U_0 that will lead to the correct formula, by preserving the Pingala recursion for the second term:

$$U_0 + U_1 = U_2$$

$$U_0 = U_2 - U_1$$

$$U_0 = 2 - 1 = 1$$

Therefore we can use these terms in our general formula, equation (4):

$$U_m = \frac{(1 - \psi)\varphi^m + (\varphi - 1)\psi^m}{\sqrt{5}}$$

$$U_m = \frac{\varphi^m - \psi\varphi^m + \varphi\psi^m - \psi^m}{\sqrt{5}}$$

$$U_m = \frac{\varphi^m - \psi\phi\varphi^{m-1} + \varphi\psi\psi^{m-1} - \psi^m}{\sqrt{5}}$$

Using the exact values of φ and ψ, we can see that:

$$\varphi\psi = \left(\frac{1 + \sqrt{5}}{2}\right)\left(\frac{1 - \sqrt{5}}{2}\right)$$

$$\varphi\psi = \frac{1 + \sqrt{5} - \sqrt{5} - 5}{4}$$

$$\varphi\psi = -1$$

This allows us to simplify our expression for U_m:

$$U_m = \frac{\varphi^m + \varphi^{m-1} - \psi^{m-1} - \psi^m}{\sqrt{5}}$$

And using the Pingala recursion on each pair:

$$U_m = \frac{\varphi^{m+1} - \psi^{m+1}}{\sqrt{5}}$$

Compared to equation (5), this satisfies the relationship that $U_m = F_{m+1}$, which aligns with our previous conclusion, suggesting we have found the correct formula. However, you may still be unconvinced by this formula. How can the Fibonacci series, such a simple recursive sequence on integers, need such a complicated formula to model it? Surely we do not need three different irrational numbers [1], φ, ψ and $\sqrt{5}$, just to find a closed-form formula for these numbers? So, to persuade you, these are the first 16 Fibonacci numbers calculated with our formula:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610

Fig. 5: The Fibonacci numbers calculated with the closed-form formula are exactly equal to those calculated with the Pingala recursion.

Conclusion:

I find this formula a very beautiful result, due to the golden ratio, a famous mathematical constant, arising out of a seemingly simple and unrelated sequence. It is a great example of why mathematicians study abstract-seeming concepts like irrational numbers, though infinite precision is not possible in the real world, and why they try to generalise concepts. Using these ideas, we have been able to solve an ancient puzzle about poetry – a brilliant demonstration of the power and interconnectedness of mathematics.

I would like to leave you with a challenge. We have studied matra-vrttas with two types of syllable, single morae and double morae syllables. If there were also syllables containing three morae (which do not actually exist in Sanskrit), this would give us new ways of arranging a line. Can you figure out how many ways there are to arrange a line of m morae, where syllables can contain one, two or three morae? To check your answer, research the 'Tribonacci' numbers! [2]

[1] An irrational number is a number that cannot be written in the form: $\frac{a}{b}$ where both a and b are integers. φ, ψ and $\sqrt{5}$ are all examples of these.

[2] As an interesting side-note, Fibonacci originally invented the Fibonacci sequence to solve a puzzle simulating the population of rabbits in a field, which is surprisingly connected to Pingala's poetry. Much in the same way, the Tribonacci numbers were first invented to model elephant populations, by none other than Charles Darwin! (2) He was not a mathematician, and so the connections to the Fibonacci numbers were not of interest to him, however retrospective research has highlighted this interesting connection to Fibonacci, and indeed to our modified Pingala poetry puzzle. The first person to formally study the Tribonacci numbers, to our knowledge, was a Belgian mathematician called Agronomof (3).

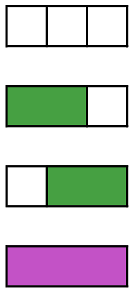

Fig. 6: All the ways of arranging one 3-morae matra-vttra line where triple-morae syllables exist. There are more conceivable arrangements than where triple-morae syllables do not exist.

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