The Golden Series

1.1 Introducing the Fibonacci Series

Imagine you are at the bottom of the staircase. Given that you can only climb the stairs 1 or 2 at a time, in how many ways can you reach the top? Let us examine the first few results:

Number of stairs climbed (n)	0	1	2	3	4	5	6
Ways of climbing	-	1	1,2	111,12,21	1111,112,121,211,22	11111, 1112, 1121, 1211, 2111, 221, 212, 122	111111, 11112, 11121, 11211, 12111, 21111, 2211, 2121, 2112, 1212, 1221, 1122, 222
Number of ways (x)	1	1	2	3	5	8	13

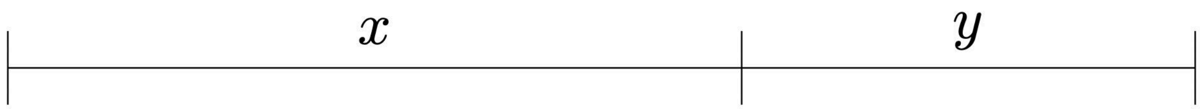
You might notice that the numbers in the bottom row follow a pattern - the number of ways of climbing n+1 stairs seems to be equal to the number of ways of climbing n stairs plus the number of ways of climbing n-1 stairs. To justify this, we might say that to climb n+1 stairs, we can climb 1 step with our first step, in which case the number of ways of climbing n+1 stairs, $x_{n+1(i)}$, = the number of ways of climbing n stairs, x_n. Alternatively, we can climb 2 stairs with our first step, in which case $x_{n+1(ii)}$ = the number of ways of climbing n-1 stairs, x_{n-1}. We then sum the two equations to get:

$$x_{n+1} = x_n + x_{n-1}$$

This relationship dictates the recursive relation in what is considered as one of the most famous sequences in mathematics - the Fibonacci sequence. In this essay I hope to shine the golden light on many of its seemingly mysterious characteristics, including the prevalence of the Fibonacci numbers in nature.

1.2 The Golden Ratio

We will now derive the value of a special constant linked to the Fibonacci sequence. Take a line, and split it into two sections of length x and y, such that the ratio of the larger to the smaller section is equal to the ratio of the length of the whole line to the larger section.



Thus, $\frac{x}{y} = \frac{x+y}{x}$

Splitting the fraction, $\frac{x}{y} = 1 + \frac{y}{x}$

Now, let $\frac{x}{y} = \phi$. This implies that: $\phi = 1 + \frac{1}{\phi}$

Which gives us the quadratic: $\phi^2 - \phi - 1 = 0$

The positive root of which is $\phi = \frac{\sqrt{5}+1}{2} \simeq 1.618$. The negative root of course does not make sense, as we are taking the ratio of two positive lengths. This constant has a special name - the golden ratio.

But how does this relate to the Fibonacci sequence? Suppose we want to get an approximation for the nth Fibonacci Number. We know that

$$F_{n+1} = F_n + F_{n-1}$$

Dividing by F_n: $\frac{F_{n+1}}{F_n} = 1 + \frac{F_{n-1}}{F_n}$

If we assume that $\frac{F_{n+1}}{F_n}$ converges to a limit L for large n, then $\frac{F_{n-1}}{F_n}$ equally converges to $\frac{1}{L}$. The equation now becomes: $L = 1 + \frac{1}{L}$

This is the quadratic we saw earlier, with positive solution ϕ - the golden ratio! We can therefore deduce that the ratio between consecutive fibonacci numbers approaches ϕ, which gives us the useful approximation that the nth Fibonacci number $\simeq \phi^n$ for large n. In fact, there is also an exact formula for the nth Fibonacci number:

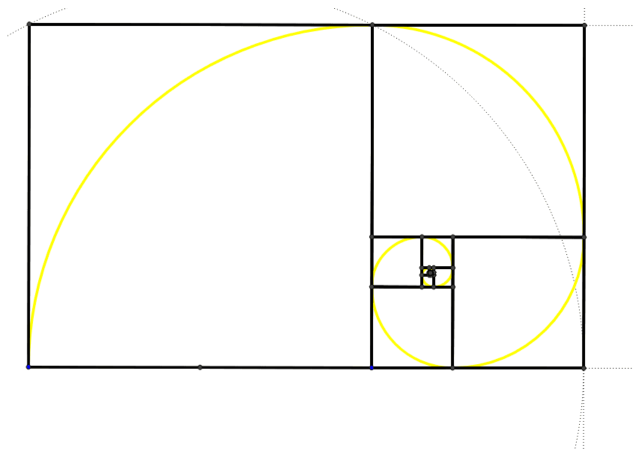
$$F_n = \frac{\phi^n - (-\varphi)^n}{\sqrt{5}}$$ ($\varphi = \phi - 1 = \frac{1}{\phi}$, the reciprocal of the golden ratio).

This is known as Binet's formula, and you could even prove it yourself by induction if you like.

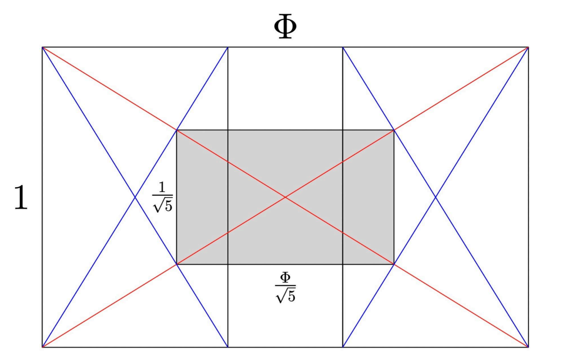
2.1 The Golden Spiral

Now that we have laid the foundations, it is time to see one of the beautiful visualisations of this sequence. We start by constructing the 'golden rectangle' - a rectangle with length Φ and width

1. Then, we construct a 1x1 square in the left portion of the rectangle, splitting the total shape into a square and another rectangle with length 1 and width Φ-1, or φ. It turns out that this new rectangle is similar to the large rectangle, since $\frac{\phi}{1} = \frac{1}{\varphi}$. Therefore we repeat this process, until the rectangles and squares become too small to draw.

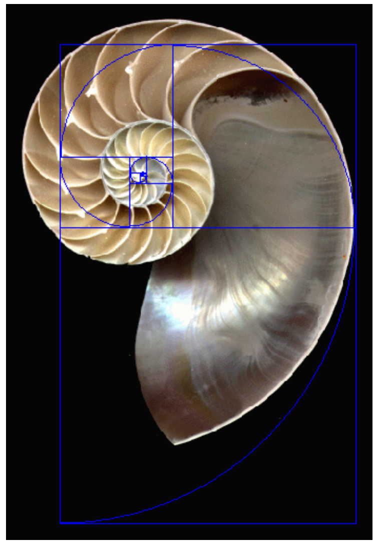


I have included a spiral, starting from the bottom left and finishing at the 'point of convergence' - where the squares meet as they get smaller and smaller. (Full marks for guessing why it is coloured in gold.) In fact, if you construct 4 copies of the golden rectangle, with this 'singularity' in the 4 different corners of the rectangle, then join these 4 points up to create a new rectangle-



-this new rectangle will also be a golden rectangle, just scaled down by a factor of √5 ! Pretty right? In fact, this innate beauty is reflected in its use in many 'facial attractiveness' calculators - according to them, the face is most aesthetically pleasing when the width of the face is 1.618 times the width of the mouth. The validity of this claim is still disputed, but we can now see the logic behind it.

In addition, this spiralling pattern, perhaps surprisingly, is seen frequently in nature. Take a look at this snail:



Coincidence, you might say, that its frame closely resembles the golden spiral. Admittedly, you may be right to an extent; however, there is a scientific reason for which the golden ratio resurfaces. Nevertheless, we must first embark on a mini detour to learn more about this fascinating constant…

2.2 Continued Fractions

Continued fractions are fractions of the form:

$$x = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots}}}$$

Remember that a rational number is a number that can be expressed as $\frac{p}{q}$, where p and q are coprime integers (i.e. whole numbers that do not share any factors apart from 1), with $q \neq 0$. Therefore, we deduce that rational numbers have a finite continued fraction expansion - as an example, the expansion of $\frac{2}{3}$ is as follows:

$$\frac{2}{3} = 0 + \frac{2}{3}$$
$$= 0 + \frac{1}{\frac{3}{2}}$$
$$= 0 + \frac{1}{1+\frac{1}{2}}$$

Here, $a_0, a_1, a_2 = 0, 1, 2$. Common shorthand used to denote this is $\frac{2}{3} = [0; 1, 2]$, or more generally, $x = [a_0; a_1, a_2, \dots a_n]$.

On the contrary, irrational numbers cannot be expressed in this form; it is impossible to write them as a ratio of integers. Consequently, they have an infinite continued fraction expansion. As an example:

$$\begin{aligned}\pi &= 3.141... \\ &= 3 + 0.141... \\ &= 3 + \frac{1}{7.062...} \\ &= 3 + \frac{1}{7+0.062...} \\ &= 3 + \frac{1}{7+\frac{1}{15+0.996...}}\end{aligned}$$

This gives us $\pi = [3; 7, 15, ...]$. Notice that the approximation that $\pi \simeq \frac{22}{7}$ comes from its continued fraction: $\pi \simeq 3 + \frac{1}{7}$. As such, using more terms would yield a better approximation.

So how would we go about finding the expansion for Φ? Well, remember that:

$$\phi = 1 + \frac{1}{\phi}$$

We can rewrite the Φ in the denominator as $1 + \frac{1}{\phi}$:

$$\phi = 1 + \frac{1}{1+\frac{1}{\phi}}$$

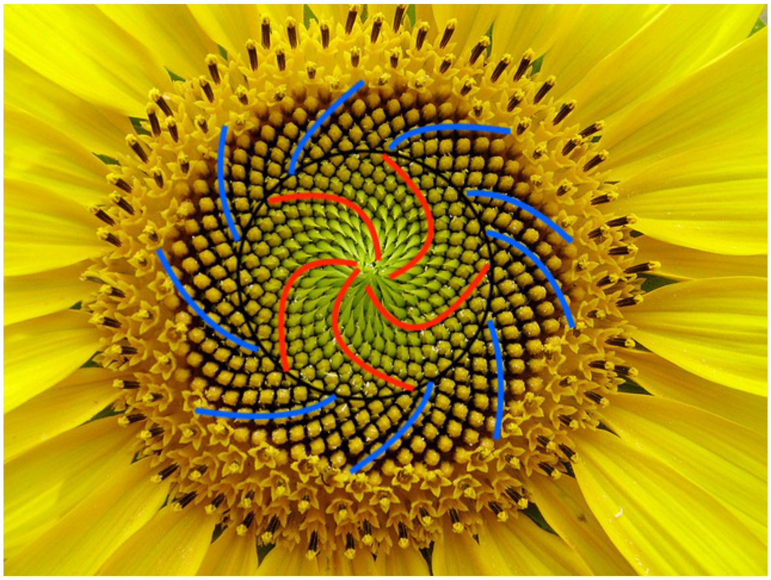
Finally, we can do the same thing as many times as we wish, which gives us the full expansion for Φ:

$$\phi = 1 + \frac{1}{1+\frac{1}{1+...}}, \text{ or } \phi = [1; \overline{1}] \text{ for short.}$$

This remarkably simple expression can be verified by taking the approximation at different points. For example, $[1; 1] = \frac{2}{1}$, $[1; 1, 1] = \frac{3}{2}$, $[1; 1, 1, 1] = \frac{5}{3}$ and so on. Spot a pattern? The fractions themselves are the ratios of consecutive Fibonacci numbers, which, as we have proved, approaches Φ as n gets large! Neat, right?

Nevertheless, the key point, which will be evident in the next section, is that it takes many terms in the continued fraction to eventually obtain a number that resembles Φ. While π can be approximated very well by just, for example, $[3; 7] \simeq 3.143$ - accurate to 2 decimal places, Φ takes a lot longer to converge - e.g. $[1; 1, 1, 1, 1] = 1.6$ - only accurate to 1 decimal place. It is quite difficult to find a rational number that accurately approximates Φ - it is for this reason that it is quite poetically coined the most irrational of the irrational numbers.

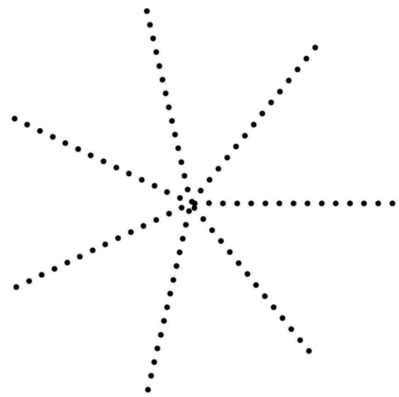
3 Fibonacci in the Sunflower



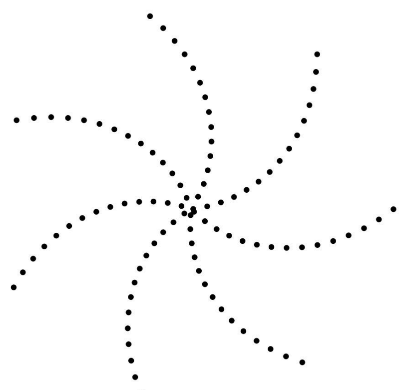
Take a look at this sunflower. Here we can see its florets in the middle, originating from the centre and being pushed outwards as more and more grow. A funny thing happens when you try and count the number of spirals, however. Have a go - count how many clockwise (coloured in blue) and anticlockwise (coloured in red) spirals of florets there are. Once you're done counting, you should find that there are 21 red spirals, and 34 blue spirals - the 6th and 7th Fibonacci numbers! Now, I know what you're thinking - that it's a coincidence once again. However, what I'd like to do now is give an explanation for why these Fibonacci numbers appear here in the centre of a sunflower, out of all places.

Let us set up a model that tries to replicate this. Suppose that each floret grows radially outwards, from the centre to the edge of the 'circle'. Also assume that each floret is rotated by some angle θ before growing. These modelling assumptions are reasonably fair to make, as this is similar to what happens in an actual sunflower. Our aim is then to figure out which angle θ optimises the distribution of florets - i.e. the angle for which the space in which the florets grow is most densely packed. Once again, we can assume that optimisation of space is an evolutionary property in most sunflowers.

We start by experimenting with a few angles. First, let's try $\Theta = \frac{1}{7} * 2\pi$ - this results in a fairly simple pattern:

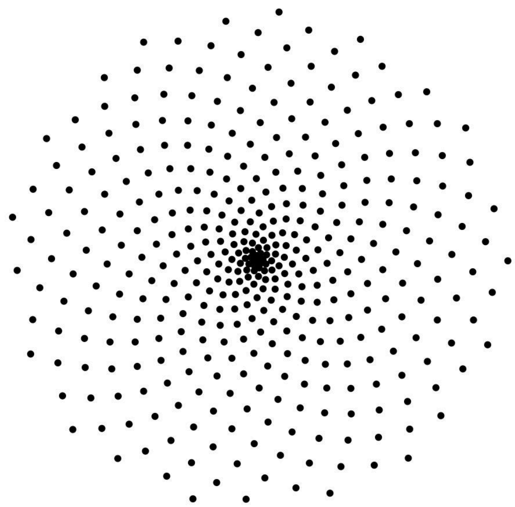


Every 7 rotations, the florets face the same direction, which results in 7 straight equally spaced lines. It follows that for every rational fraction of 2π, the pattern will be periodic - not making use of the gaps in between the lines. Maybe we should try an irrational number instead - e.g. for $\Theta = (\pi - 3) * 2\pi$:



Now we see spirals. Since $\pi - 3 \simeq \frac{1}{7}$, there are 7 of them; however, since it is not an exact approximation for $\pi - 3$, there is a slight difference. After 7 florets sprout from the centre, the direction in which the 8th is pointing is not perfectly lined with the 1st floret, which explains the curvature of the spirals.

This is where Φ comes into play. Remember that Φ is incredibly difficult to approximate as a rational fraction, especially for smaller integers. Therefore, an angle involving the golden ratio should result in a much more scattered pattern. Let's try it! We now use what is known as the golden angle - $\Theta = (1 - \varphi) * 2\pi$, where φ is the reciprocal of Φ-



-and this is the pattern we obtain. If you count the number of spirals, you should find that there are 34 clockwise and 21 anticlockwise spirals - a close resemblance to the centre of the sunflower pictured earlier! And there we have it. Optimising space is essentially the reason behind the prevalence of Fibonacci numbers in nature - it just so happens that the constant Φ is so heavily linked with the Fibonacci sequence, and simultaneously has these remarkable properties.

4 Concluding Thoughts

The beauty of the Fibonacci numbers does not end here. There are many other identities, theorems and applications of the Fibonacci sequence which I have not covered. If you are interested, the Fibonacci Q matrix and the Fibonacci bamboozlement are other great topics to look into. However, I hope to have given you some insight into the beauty behind it, as a fitting commemoration to the 13th century maths which has stood the test of time.

5 References

https://www.math.hkust.edu.hk/~machas/fibonacci.pdf
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