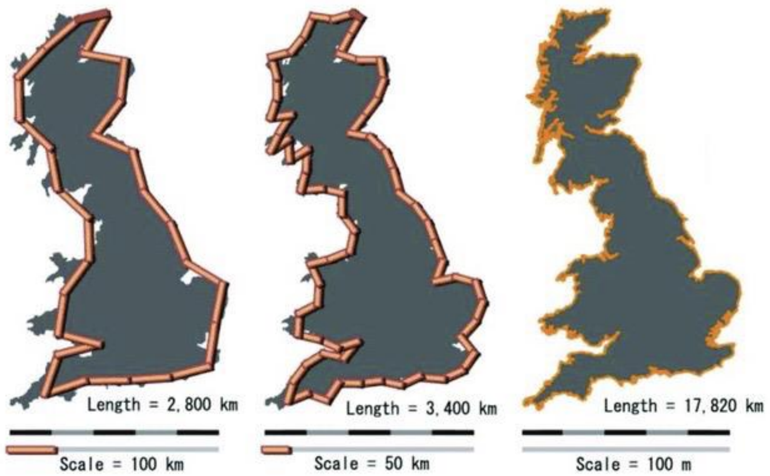
The paradoxical problem of measuring the coastline.

Although one may think measuring the coastline would be a long-forgotten puzzle, it was only in 1951 that the coastline paradox was identified by Lewis Fry Richardson. Although the problem dates back centuries, Richardson was the first to propose what became known as the Richardson effect. Just two years before his death, Richardson, whilst researching the effect that the length of the border shared between two countries had on the probability of them going to war, discovered discrepancies between different statistics. He found that the Portugal-Spain border was often quoted as either 987km or 1214km and a difference this significant sparked him to wonder what could've caused this.

At the time, the most common method for measuring borders or coastlines was to take a map or birds-eye view photograph and place x many sticks of equal length l along the border, such that each end always touched the border and started where the last stick left off. To find the border length, you would simply multiply x by l. Despite using the same method, a variation in length calculated can occur, and ultimately comes down to the length l of the stick used. This is the Richardson effect: as the length of the ruler used decreases, the overall sum is strictly increasing. He also found that, theoretically, as the measuring ruler were to approach 0, the coastline length would approach infinity.



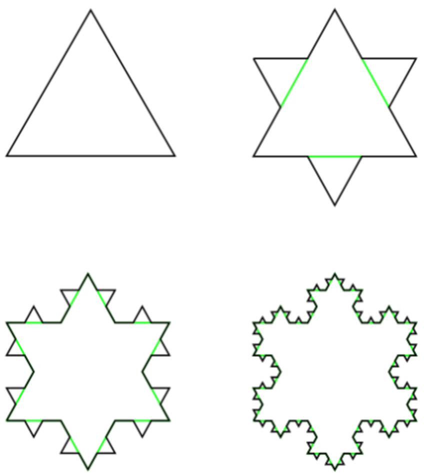
(Image from ResearchGate: https://www.researchgate.net/figure/Figure-A4-Great-Britains-coastline-paradox-from-Wikipedia-commons-and-other-cited_fig2_342144777)

But idea of something real and tangible being infinite is completely counterintuitive, so how can this be the case? Essentially this is possible as it is a limit, what the value approaches as the ruler length gets infinitesimally smaller, rather than an actual value that the coastline could be measured to. However, one would still expect the limit of something tangible like a coastline to be finite, which it paradoxically isn't.

A fractal is defined as a never ending, infinitely complex pattern, that is self repeating on any scale, meaning as you zoom in, it repeats itself forever. Whilst not technically infinitely repeating, there are many examples in nature that are thought of as taking on many fractal properties, such as: trees with branches that repeat, getting smaller and more intricate, blood vessels with complex capillaries and snowflakes. Fractals are ingrained in nature, so much so that researchers in psychology have found looking at fractals such such as trees has benefits to mental health.

Another instance of fractal properties in the natural world is coastlines. When viewed from a distance the coastline may appear simple, however when zooming in, more and more detailed indents of individual rocks can be seen. Ultimately, you could theoretically go all the way to the molecular level where there would be further detail. This can explain the Richardson effect, when a shorter length ruler is used, more of these smaller indents are accounted for which increases the length of the coastline.

The Koch snowflake is a fractal, based on the Koch curve which was first described in a 1904 paper by Helge Von Koch. It begins as an equilateral triangle of side length a, then along each edge another equilateral triangle with side length $\frac{a}{3}$ is added to the middle. After that, the same is repeated along every edge with a side length of $\frac{a}{9}$. This continues for an infinite number of iterations, producing a snowflake like shape.



(Image from wikipedia, Koch Snowflake https://en.wikipedia.org/wiki/Koch_snowflake)

The Koch snowflake can be used as an abstraction of the coastline as it has the same properties of an increasingly complex perimeter when zooming in. Therefore, by mathematically proving the Koch snowflake to have an infinite perimeter, we can assume that the same goes for the coastline of a country.

It is important to note however, that the coastline cannot in reality have an infinite perimeter, but a perimeter that tends towards infinity when measured on an infinitely increasingly detailed scale.

Before calculating the perimeter or area of the Koch snowflake, a basic understanding of geometric sequences and series is required. A geometric sequence is an ordered list of numbers where each adjacent term has a common ratio (called r) such that to get from one term to the next you must multiply it by r. A geometric sequence is often written in the form:

$u_n = a(r)^{n-1}$

where u_n is the nth term and a is the first term in the sequence.

Additionally, there is a formula that can be used to calculate the sum of every term to infinity (S_∞) of a geometric series when -1 < r < 1 (it only works when -1 < r < 1 because that is when the sum of a geometric series converges to a single value, r ≥ 1or r ≤ -1 then it will diverge to infinity). This formula is:

$S_\infty = \frac{a}{1-r}$

To show that the Koch snowflake has an infinite perimeter, we'll begin at the initial equilateral triangle of side length denoted by the constant a. By considering only one side of the triangle, the edge can be split into 3 equal sections of length $\frac{a}{3}$. When the first iteration occurs, the middle third is removed, and replaced by 2 additional edges each of length $\frac{a}{3}$, so the edge is now length $\frac{4a}{3}$. The same occurs on each side of the triangle so overall the perimeter is multiplied by $\frac{4}{3}$ and the same process occurs to each side length on each iteration.

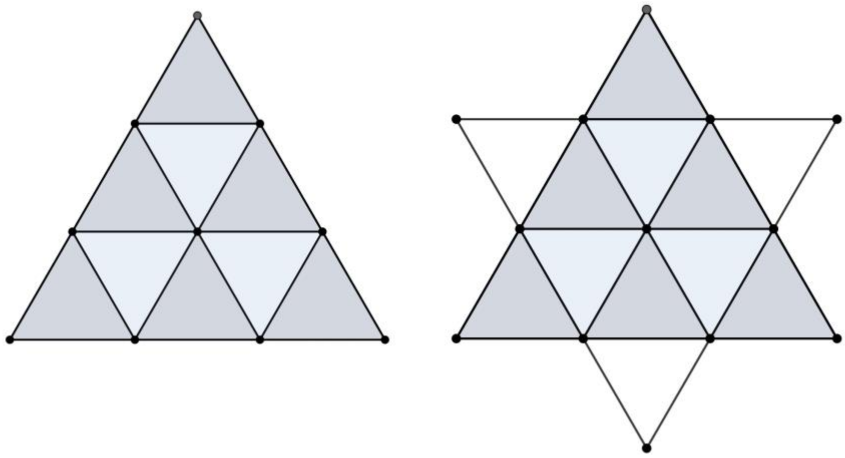
Therefore, the perimeter of the shape can be denoted by this equation where u_n is the perimeter on the nth iteration.

$u_n = a\left(\frac{4}{3}\right)^{n-1}$

This can be classified a geometric sequence where r > 1. This means as n tends towards infinity, so will the perimeter i.e. $a * \frac{4}{3} * \frac{4}{3} * \frac{4}{3} * \frac{4}{3}$ ... for an infinite number of times will be infinity. Therefore, the Koch snowflake has an infinite perimeter.

Similarly, a geometric series can be used to calculate the area of the Koch snowflake with initial area of the equilateral triangle a=1. In this case u_n will refer to the added area each iteration.

To begin, we split the initial equilateral triangle into 9 equal smaller triangles with area $\frac{1}{9}$, and on doing this is it clear that one of these is the same as the area of the triangles added in the next iteration.



This is the case because the added triangles have side lengths $\frac{1}{3}$ of the original side length so will have area $\frac{1}{3}^2 = \frac{1}{9}$. This means the added area for our first term = 3 * $\frac{1}{3}^2 = \frac{1}{3}$. Therefore a = $\frac{1}{3}$.

For the next iteration, the triangles added will have area $\frac{1}{9}^2$ (area of each triangle in the previous iteration * $\frac{1}{9}$). However, whilst in the first iteration there were 3 triangles added because the shape had 3 sides, there are now sides so 12 triangles added, so there are 4 times as many triangles to add. This means the added area this iteration will be the added area in the previous iteration * $\frac{4}{9}$. Therefore r = $\frac{4}{9}$.

With this, the geometric series:

$u_n = (\frac{1}{3})\left(\frac{4}{9}\right)^{n-1}$

can be formed to represent the added area on each iteration.

For the area of the final k snowflake, we need the initial area (1) + the sum of all the added area to infinity, which is equal to $\frac{a}{1-r} = \frac{\frac{1}{3}}{1-\left(\frac{4}{9}\right)} = \frac{3}{5}$.

So, the overall area is equal to $\frac{8}{5}$.

This demonstrates that the Koch snowflake has a finite area, yet an infinite perimeter, therefore showing that a shape like this is able to exist, so the coastline of a country can tend towards infinity, whilst it has a finite area.

To this day different organisations disagree on the length of coastlines, with the CIA World FactBook quoting Britains coastline to 12,429km, whereas ordinance survey claim it is 17,820km.

Ultimately, the problem of inconsistent statistics for the length of coastlines cannot really be solved, but does it need to be? Unless, like Richardson, you're attempting to do complex maths with border lengths, it really doesn't matter as long as the measurement is always to a suitable degree of accuracy. For the foreseeable future nobody is passionate enough to attempt to unify all organisations to use the same scale and ruler length so we shall continue to live with the inconsistencies.

<u>Bibliography</u>

Wikipedia. Coastline paradox (accessed March 2024)
https://en.wikipedia.org/w/index.php?title=Coastline_paradox&oldid=1210667801

Wikipedia. Lewis Fry Richardson (accessed March 2024)
https://en.wikipedia.org/w/index.php?title=Lewis_Fry_Richardson&oldid=1188004011

Wikipedia. Koch Snowflake (accessed March 2024)
https://en.wikipedia.org/w/index.php?title=Koch_snowflake&oldid=1195655145

Numberphile. "Measuring Coastline – Numberphile", Youtube, uploaded 27/3/2014,
https://www.youtube.com/watch?v=7dcDuVyzb8Y