The peculiarity of the World of Spheres

Introduction

Every night, if the weather treats you nice and you live in the countryside, step outside and you would see a hemisphere of stars and galaxies lighting up the whole sky. Go to your local library and you will see flat atlases of the same night sky, and flat atlases of our approximately spherical Earth. Why do flight routes look distorted on a flat map? Have you ever stopped and wondered about the works by great mathematicians behind the scenes in gathering all the knowledge necessary to understand the world of spherical surfaces? The following is a small glimpse into this strange world, The world of spherical trigonometry.

Euclidean Geometry

When you hear the word “geometry”, many of you might think of 2D shapes and their properties. Euclidean geometry is the study of space on a flat plane. It is a very intuitive subject for most people. All theorems are said to be able to be reduced to these 5 postulates:

- *Two distinct points can be connected by a straight line,*
- *any straight line can be extended indefinitely in a straight line,*
- *a circle can be produced given a radius and a centre,*
- *all right angles are equal to each other,*
- **deep breath* when two straight lines intersect another straight line, if the sum of the two interior angles produced is less than 180°, the two straight lines would intersect in the same side when extended indefinitely, and would diverge away on the other side of the “another straight line”.*

As you might have noticed, the fifth postulate, also known as the “parallel postulate” seems like an odd one out. The first four seems obvious but not the fifth postulate. In which case, congratulations, you think alike as great mathematicians as Gauss and many others. Due to this idea that the parallel postulate looks suspicious, a few simpler postulates have been produced that are equivalent to this postulate. A few non-equivalent postulates have also been made. These non-equivalent postulates resulted in the birth of non-euclidean geometry.

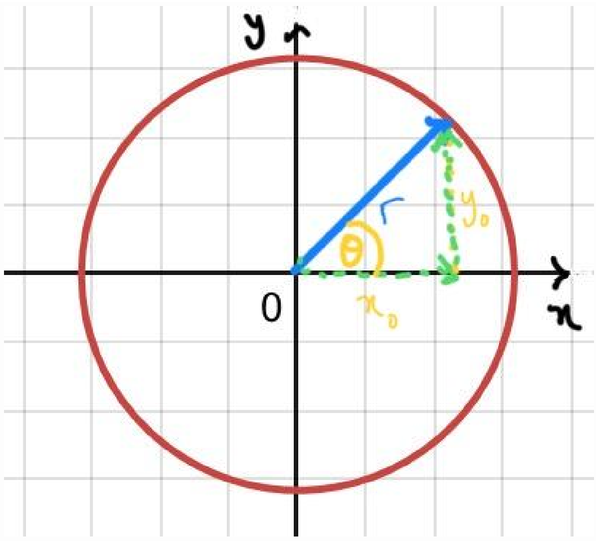
It can be understood and implied from the fifth postulate, that when the sum of the interior angles equal to 180°, the two straight lines neither diverge nor converge. These two straight lines are said to be “parallel”.

It was hypothesised that one can make other geometric models that disobey the parallel postulate, but keep the first four. Thus was the beginning of the exploration by mathematicians in the unintuitive world of non-euclidean space.

Imagining the Surface of a Sphere

On a space that obeys the Euclidean postulates, a way to describe lines mathematically is by the cartesian system. In such a system, there are standard axes, usually denoted by variables 'x' and 'y' in two dimensional space. The letter 'z' is used for the third direction in three dimensional space. These lines represent where the corresponding coordinate equals to zero. Points on one side of the axes have the corresponding variable (x, y or z) coordinate going up as the point is further away. On the opposite side and direction of the axes, the coordinate goes down instead. Lines and curves can be drawn with equations in terms of x, y and z.

For more complex objects to be described, parametric equations are used. In such a system, each x, y and z are written in parameters in terms of t. For example, let's think about the description of a circle on an xy-plane. A circle is just a locus of points that is a certain distance from a particular point. In this example, let's set this "particular point" as the origin, (0,0), and the "certain distance" to be r, which corresponds to the radius of the circle.



Extend a line from the origin to a point on the circle. Let the x-coordinate and the y-coordinate of this point be x_0 and y_0 respectively. And also let the angle this line makes with the positive x-axis be of the size $\theta°$.

A right triangle is produced, with the hypotenuse as the line with distance r. The following facts are thus known:

$\cos(\theta°) = x_0 / r$
$\sin(\theta°) = y_0 / r$

These can be rearranged to:

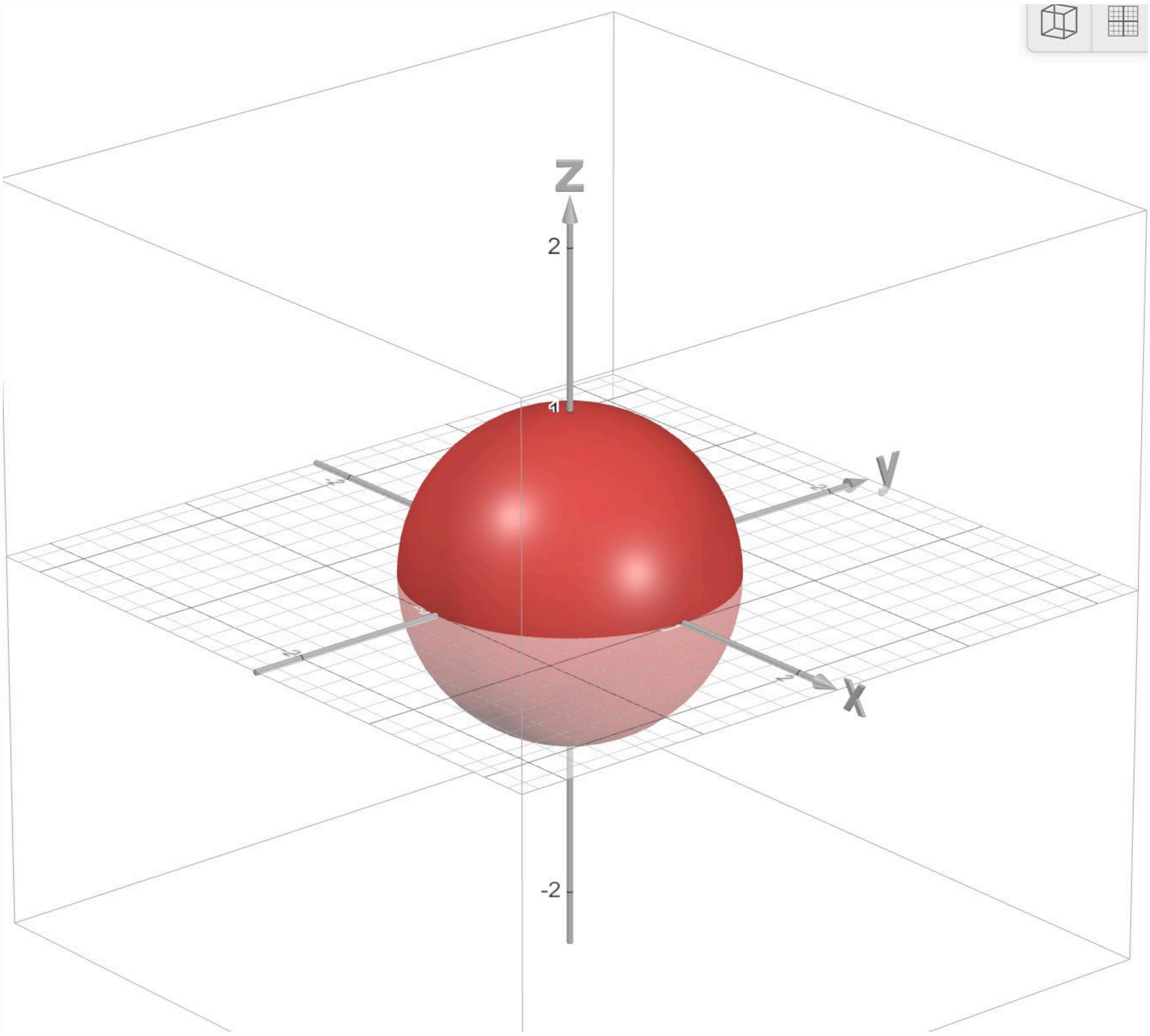
$x_0 = r \cos(\theta°)$
$y_0 = r \sin(\theta°)$

Therefore, more generally, a circle can be written as the parametric equations:

$x = r \cos(t)$
$y = r \sin(t)$

To map out objects on a spherical plane, we can use the idea of the cartesian system with a few twists. Firstly, let us map the plane itself in a 3 dimensional cartesian space. The plane can be defined as the surface of a sphere. Let's imagine this sphere mathematically by considering its equation.

A sphere, just like a circle, is a locus of points that is a certain distance to a fixed point.



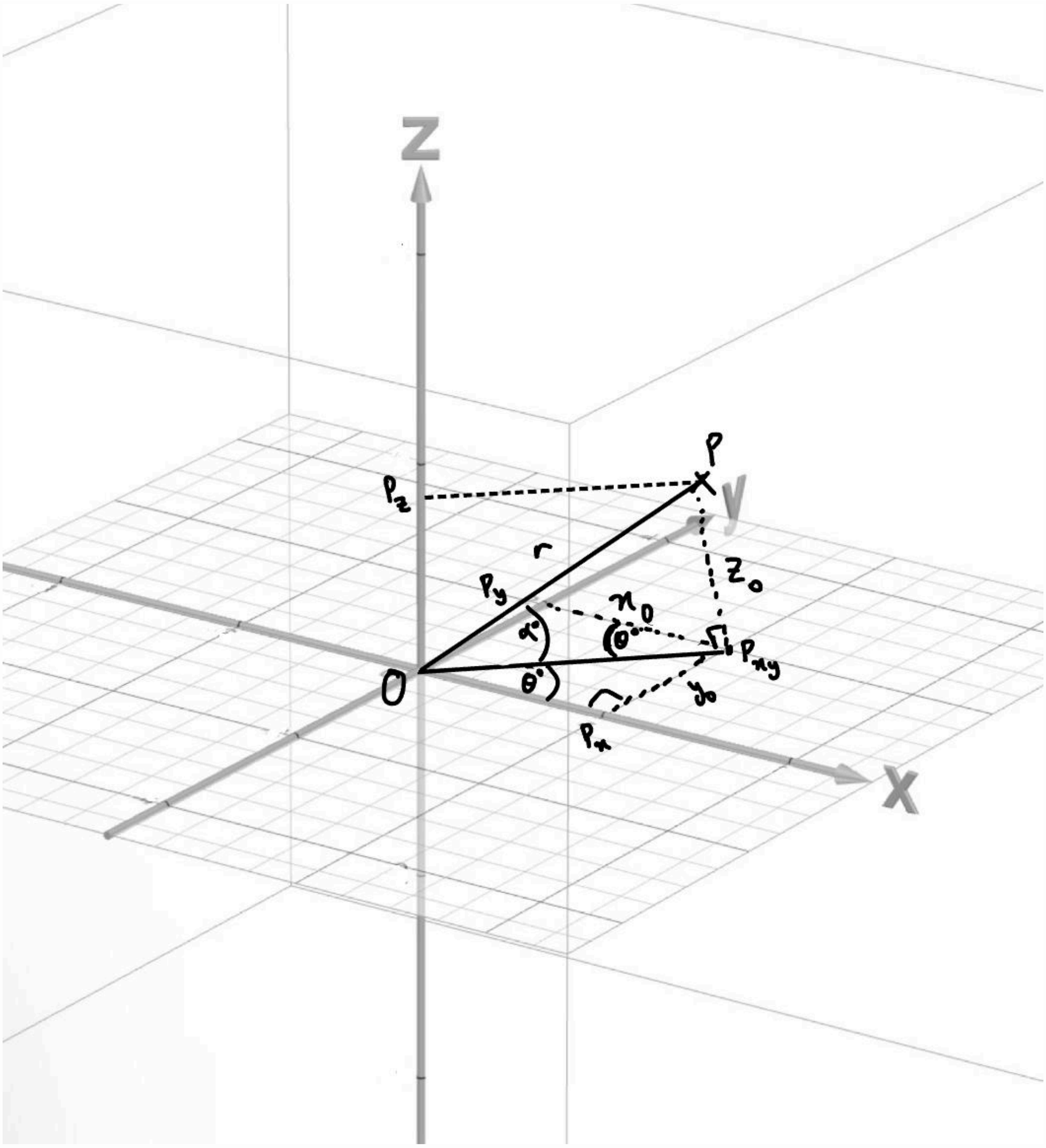
Please note that the plane only consists of the surface of this sphere. We are not interested in the "filling" of the sphere.

A straightforward way of thinking to describe a point on the surface of a sphere would be using the cartesian coordinates. However, after some thought, although it is possible, it would not be very practical or helpful. This is because it takes quite a bit of effort to determine which points are on the surface of the sphere.

Instead, we will take into advantage the fact that all of these points are of the same distance to the centre. And each of these points can be written in parametric equations in terms of two angles.

Let's explore this by extending a line from the centre of a circle to a point on the boundary of the sphere. Let the centre of the sphere be O. Let the radius of the sphere be r. Let the point of intersection between the line and the boundary be $P(x_0 , y_0 , z_0)$. Extend a line, parallel to the z-axis, from point P. Let the point on this line where z = 0 be P_{xy}. When a line parallel to the y-axis is extended from P_{xy}, let the point of intersection of this line and the x-axis be P_x. When a line parallel to the x-axis is extended from P_{xy}, let the point of intersection of this line and the y-axis be P_y.

And let angle POP_{xy} be $\alpha°$. Also let angle P_xOP_{xy} be $\theta°$. Since angle $P_yP_{xy}P_x$ is a right angle, and angle $OP_{xy}P_x$ is $90° - \theta°$, angle $P_yP_{xy}O$ is also $\theta°$.



Multiple right angled triangles appear. The three relevant ones are triangle P_xOP_{xy}, triangle POP_{xy} and triangle P_yOP_{xy}. Expressions of x_0, y_0, and z_0 can be written in terms of r, θ and α. To do so for x_0 and y_0, expression of OP_{xy} in terms of r, θ and α needs to first be determined.

Focusing on triangle POP_{xy},

$\cos(\alpha°) = OP_{xy} / r$

So, $OP_{xy} = r\cos(\alpha°)$

And we also know that:

$\cos(\theta°) = x_0 / OP_{xy}$

$\sin(\theta°) = y_0 / OP_{xy}$

$\sin(\alpha°) = z_0 / r$

Therefore, the coordinates on a circle can be determined from information of r, α° and θ° by:
$x = r \cos(\alpha°) \cos(\theta°)$
$y = r \cos(\alpha°) \sin(\theta°)$
$z = r \sin(\alpha°)$

We should be aware that r is constant on the same spherical surface. Only α and θ change with different points on the surface. And that by thinking of (0, 0, r) as the north pole, and (r, 0, 0) as the point on the equator of the prime meridian, α is the latitude, while θ is the longitude.

Now, let's make a line on a sphere's surface. By Euclid's first postulate, let's consider two points on this plane. M_1 and M_2. A straight line can then be drawn as the shortest path between these two points. The "shortest path" might not be obvious at first. All sorts of paths can be drawn to connect the points but only one is the shortest.

By visualisation, the shortest path seems to be the line that when extended, passes through the diameter line of M_1 and M_2. The line will eventually meet the starting point. When extended fully around the sphere, the line is called the "great circle".

A peculiarity to be noticed when making many many great circles is that none of them are parallel with any other great circle, meaning that a great circle intersects with every other great circle possible on the sphere. This makes sense because looking at a great circle produced by connecting two points M_a and M_b, and now looking at possible paths taken by connecting two points M_c and M_d such that each of the pairs M_a and M_c, and M_b and M_d have the same value of α but a very tiny difference in θ. It is not possible for the connecting path of M_c and M_d to not pass through the great circle while also being the shortest path that intersects with the diameter of M_c and M_d on the opposite side.

Conclusion

I hope this has made clear how unintuitive spheres are, and the imminent vitalness that this new world is explored further. This, like all other fields of mathematics will find their way in our daily lives, us benefiting from them in many more ways than one might expect.