The joy of solving small problems

Introduction

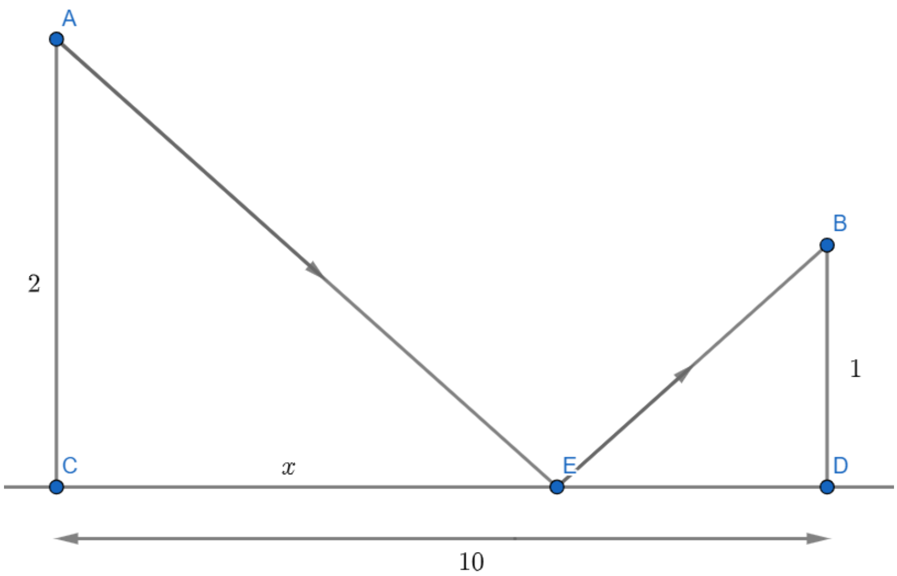
How can a tale about a worried farmer and their cow lead to elegant mathematics? How can this all link to not only different area of maths, but an area of physics? In this essay, I wish to answer both of these questions by introducing a seemingly innocent problem that yields great rewards to whoever chooses to investigate it thoroughly.

At a deeper level, I wish to demonstrate why spending time on niche, small problems lays the groundwork for more complex thinking. But, more importantly, I wish to showcase a problem that leads to a piece of mathematics that is elegant, surprising and beautiful all at once.

The problem at hand

Imagine a farmer, stood in a field at a point A, trying to save an injured cow at located at a point B, elsewhere in the field. Usually, the cow would be able to drink from a river that flows through the field, but is unable to today. To help the poor cow, the farmer must travel from their position to the edge of the river, fill up a bucket of water and then travel to the cow. To keep the cow as comfortable as possible, the farmer must complete this task as quickly as possible.

The farmer is stood 2 metres away from the edge of the river at one end; the cow is lying 1 metre away from the water's edge at the other end; the river flows in a straight line for 10 metres.



Assuming the farmer travels at a constant speed for the entire journey, at what distance x along the river should they go to fill up their bucket to reach the cow in the shortest time?

In short, what value of x will result in the smallest distance $AE + EB$?

This problem belongs to a class of problems known as optimisation problems. These involve situations where you need to find the 'best', or optimal, course of action. Here, the farmer could take many different paths to the cow, but we want to find the one that takes the smallest amount of time to travel.

Cold, hard calculus

Many optimisation problems like this one can be tackled using calculus – a branch of mathematics that studies the rate of change of a variable quantity.

However, to use calculus, we must first link the total distance travelled to the farmer – which I will refer to as D – to the length x. This can be done using Pythagoras's theorem because the triangles ACE and BDE are both right-angled.

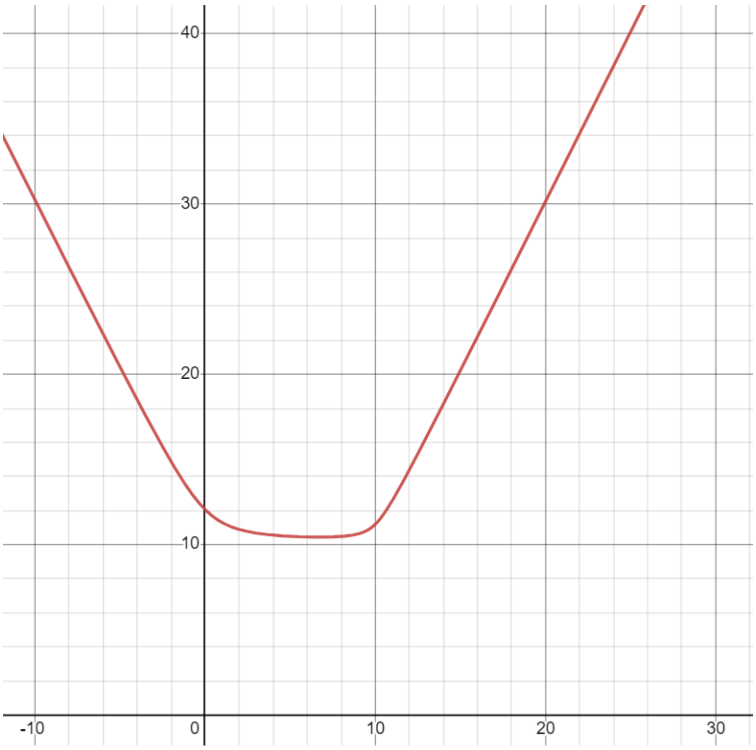
This produces the following equation for D in terms of x:

$$D = \sqrt{2^2 + x^2} + \sqrt{1^2 + (10 - x)^2}$$

Which can be simplified to:

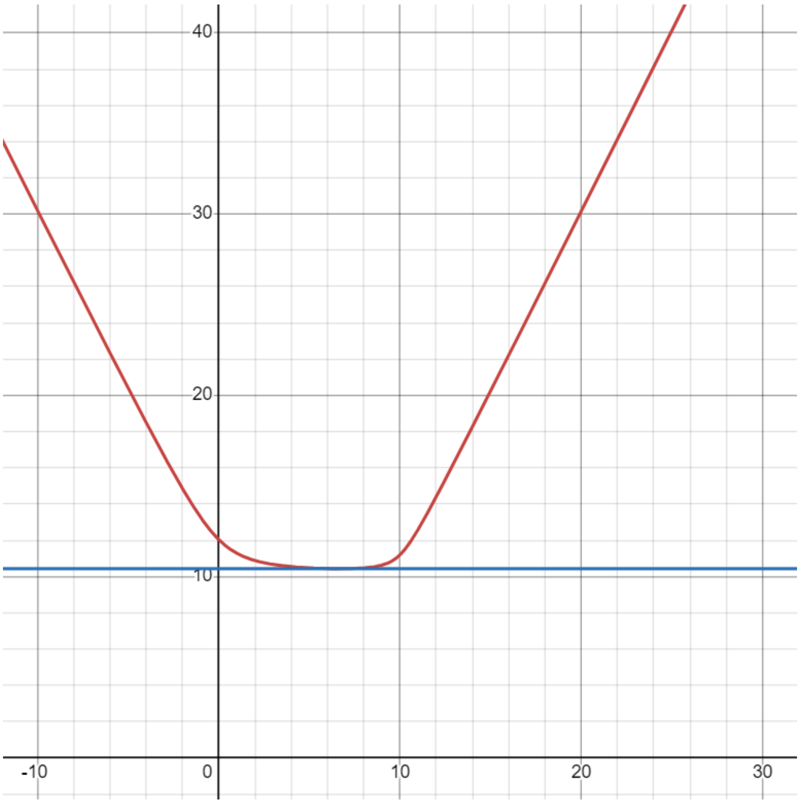
$$D = \sqrt{x^2 + 4} + \sqrt{x^2 - 20x + 101}$$

We can then use this equation to plot a graph of D against x, as shown below.



We can see by inspecting the graph that the minimum distance is just over 10 metres, and it occurs when x is somewhere between 6 and 8. The graphing package I am using actually tells me that the minimum value of D is 10.44, and it occurs when $x = 6.667$ (I assume these numbers have been rounded to 4 significant figures), so I suppose any normal person would stop there. But using a graphing package to calculate the minimum value for you not only ruins the fun, but does not teach you how to do the maths by yourself.

The smallest value of a graph is known as a minimum point. Actually, a minimum point does not have to be the smallest point on the entire graph; instead, it just has to be the bottom of a small dip in the graph, before it starts going back up. A minimum point is a type of stationary point, which occur when the gradient of the graph is equal to 0. This can be seen more clearly by looking at the graph below, where I have added the tangent to the curve at this minimum point (the blue line). As you can see, that the tangent to the curve is perfectly horizontal and, thus, has a gradient of 0.



View the graph here: https://www.desmos.com/calculator/ravj6t8ovy

Now, all we need is to find where the gradient of this graph is equal to zero, which is a classic application of calculus. To find the gradient of this line at any point along it, we can differentiate it with respect to x.

Explaining differentiation is beyond the scope of this essay, but by using the chain rule, the following expression can be deduced for the differentiation of D with respect to x:

$$\frac{dD}{dx} = \frac{1}{2}(2x)(x^2+4)^{-\frac{1}{2}} + \frac{1}{2}(2x-20)(x^2-20x+101)^{-\frac{1}{2}}$$

Which can be simplified to:

$$\frac{dD}{dx} = \frac{x}{\sqrt{x^2+4}} + \frac{x-10}{\sqrt{x^2-20x+101}}$$

We want to know the value of x for which $\frac{dD}{dx}$ is equal to 0. This translates to the following:

$$\frac{x}{\sqrt{x^2+4}}+\frac{x-10}{\sqrt{x^2-20x+101}}=0$$

$$\Longrightarrow \frac{x-10}{\sqrt{x^2-20x+101}}=-\frac{x}{\sqrt{x^2+4}}$$

The square roots in the denominators of the above fractions are a bit of a pain. The easiest way to get rid of them would be to square both sides of this equation. It is not always the best idea to do this because it can lead to extraneous solutions due to information being lost about the sign of each term. To illustrate this point, consider the following workings:

$$x=5$$

$$x^2=25$$

$$\Rightarrow x=\pm 5$$

As a result of squaring and then square rooting both sides of the (insultingly simple) equation $x=5$, we introduced $x=-5$ as a solution that is clearly not a solution to the original equation.

Having said this, it is alright to square both sides of an equation, as long as you are aware of the implications of doing so. This is exactly what we will do now.

$$\frac{(x-10)^2}{x^2-20x+101}=\frac{x^2}{x^2+4}$$

Cross-multiplying and simplifying produces the following:

$$(x-10)^2(x^2+4)=x^2(x^2-20x+101)$$

$$x^4-20x^3+104x^2-80x+400=x^4-20x^3+101x^2$$

As you can see, both the x^4 and x^3 terms will cancel out, leaving a quadratic equation behind:

$$3x^2-80x+400=0$$

$$(x-20)(3x-20)=0$$

$$x=20, \qquad x=\frac{20}{3}$$

A bit of thought reveals that $x=20$ is not a sensible solution because the river is only 10 metres long; one might question the sanity of the farmer if they missed the river entirely. In fact, it is an extraneous solution that was introduced after squaring both sides of the equation. I told you we had to be careful!

Substituting the actual solution $x = \frac{20}{3}$ into the original equation for D gives us the total distance the farmer needs to travel:

$$x = \frac{20}{3} \approx 6.667$$

$$D = \sqrt{109} \approx 10.44$$

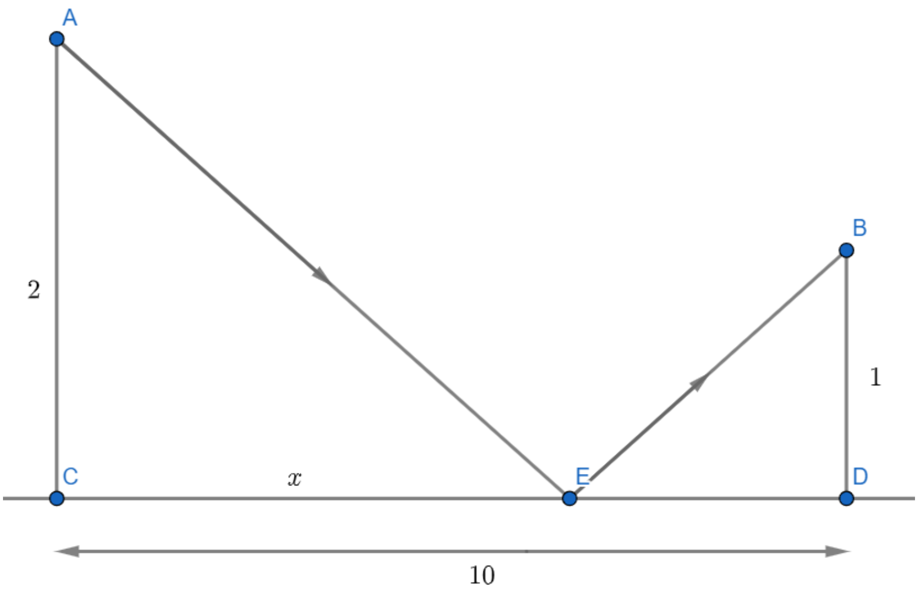
This matches the solution given by the graphing package I used earlier, which is always reassuring; it would have been very embarrassing if it did not. Furthermore, splitting the length of the river ($10m$) into the ratio of the lengths $AC : BD$ yields $\frac{20}{3} : \frac{10}{3}$. This seems to make intuitive sense that the lengths CE and ED should be in the same ratio as the distances AC and BD.

While this method results in the correct answer, I would not describe it as elegant. Elegance requires simplicity and, while the techniques used in this solution are taught to any student studying A-level maths, it would be terribly difficult to explain it to anybody with little mathematical experience. However, the main issue I have with this approach is that, beneath all of this, it simply feels like there should be an easier method…

A more elegant solution

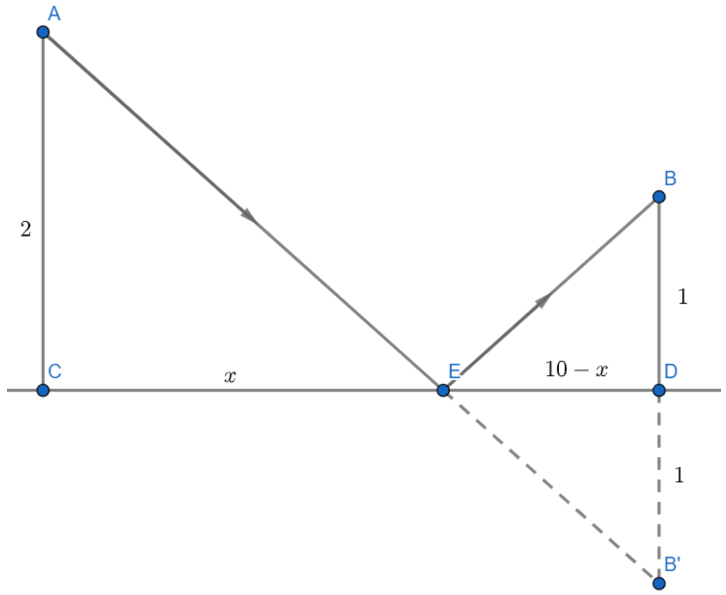
Let's temporarily forget what we have just done and, instead, take a step back and think about the problem before immediately diving into a solution.

Below is a repetition of the diagram shown previously.



With the above arrangement, the shortest path from A to E to B is not immediately obvious. Could the problem be altered to make this shortest path easier to find?

If we reflect the point B to the other side of the river, the shortest path that should be taken by the farmer becomes trivial! In this new arrangement of points, the shortest path between A and the reflected point (which I will label B') is the *straight-line* distance between them.



We have now changed the original optimisation problem into a simpler geometry one. I encourage you to try and find the length x in this new situation where AB' is a straight line.

The solution that I will show relies on triangles ACE and $B'DE$ being similar. We know this because lines AC and DB' are both perpendicular to CED and are, thus, parallel. Therefore, $\angle CAE$ and $\angle EB'D$ are alternate and must be equal; $\angle CEA = \angle B'ED$ because CED and AEB' are both straight lines; finally, $\angle EDB' = \angle ECA = 90°$. Both triangles have the same angles, so are scaled up or scaled down versions of each other.

If two triangles are similar, their corresponding sides lie in the same proportion. From this, we can deduce the following statements:

$$\frac{ED}{DB'} = \frac{CE}{AC}$$

$$\Rightarrow \frac{x}{2} = \frac{10 - x}{1}$$

Solving this equation gives our answer for the value of x:

$$x = \frac{20}{3}$$

The reason that I am so fond of this second approach is that it shows how clever thinking can sidestep pages of algebra. Not only is this solution much easier to follow, but it feels far more satisfactory: it is without the extraneous $x = 20$ solution that was disregarded previously, and there is a wonderful sense of familiarity associated with the use of similar triangles, which appear in so many geometry problems.

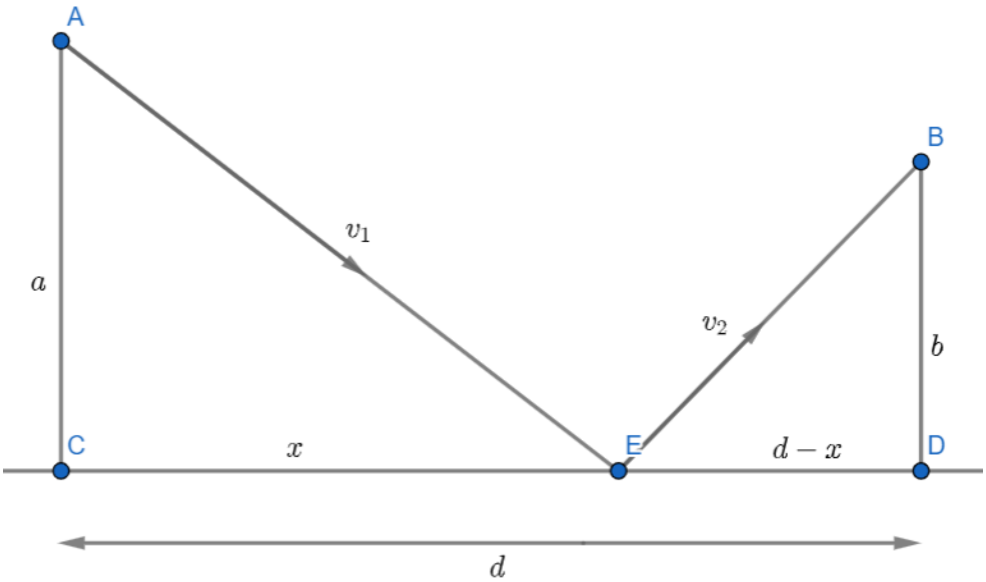
I believe that elegant solutions like these are responsible for the joy that many people experience from solving maths problems.

So, the natural question is: which solution is better? While this geometrical approach is, undeniably, the most elegant for this particular problem, we should not be so quick to disregard the calculus seen in the first approach. It is more robust and does not rely on any underlying symmetry. The problem that I will show next highlights this well.

A harder problem

In the previous problem, the farmer travelled at a constant speed for the entire journey. If we want this model to be more realistic, we must account for the fact that the farmer will travel more slowly after filling up their bucket because of the added weight of the water in the bucket. Let the farmer's velocity on the way to the river be v_1 and their velocity on the way back be v_2.

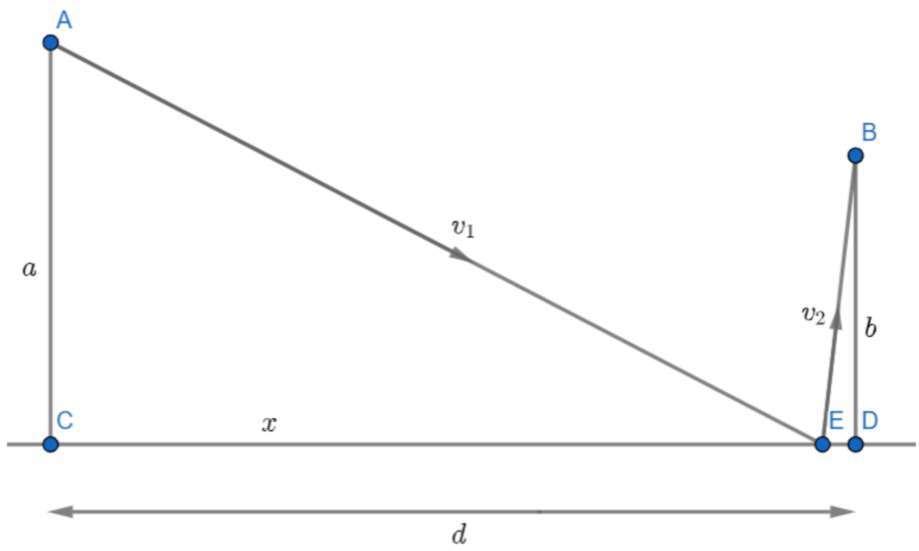
Let's also generalise the problem: rather than the farmer and cow being specific distances from the river, they are now located a and b metres from the water's edge respectively. The horizontal distance between the farmer and cow (the length of the river) is now d metres.



In this generalised problem, where the farmer's velocity changes, at what distance x along the river should the farmer fill their bucket so that they can reach the cow in the smallest amount of time?

The calculus strikes back

The trick of reflecting point B will not work here because the path that the farmer should take is not necessarily a straight line from A to the reflected point B'. While this would be the shortest distance, it will not necessarily be the path that takes the farmer the least amount of time to travel. This makes more sense if you imagine the farmer carrying a comically heavy bucket of water, making them travel very slowly after filling it up. In this instance, they should take a path similar to the one below, which reduces the distance they must travel while carrying the full bucket.



This means that, whether you like it or not, we are back to using calculus.

Splitting the journey into two sections (A to E, and E to B) makes the problem easier to solve because the farmer will be travelling at a constant velocity in each section.

Using the classic equation $speed = distance\ /\ time$, we can deduce the following equation for the time, t, it takes for the farmer to reach the cow:

$$t = \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{b^2 + (d - x)^2}}{v_2}$$

In the same way as earlier, this can be differentiated to produce the following expression (bearing in mind that a, b and d are constants):

$$\frac{dt}{dx} = \frac{x}{v_1\sqrt{a^2 + x^2}} + \frac{x - d}{v_2\sqrt{b^2 + (d - x)^2}}$$

Attempting to find the minimum point (where $\frac{dt}{dx} = 0$) results in the following working:

$$\frac{x}{v_1\sqrt{a^2 + x^2}} = -\frac{x - d}{v_2\sqrt{b^2 + (d - x)^2}}$$

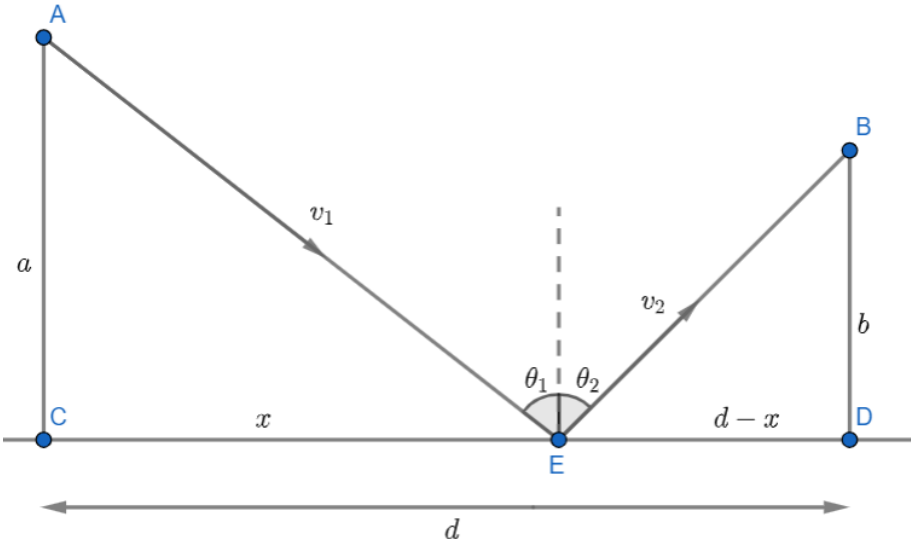
$$\therefore \frac{x}{v_1\sqrt{a^2 + x^2}} = \frac{d - x}{v_2\sqrt{b^2 + (d - x)^2}}$$

While this equation is not pretty, it is the solution to this new problem. Given the values of each of the constants a, b, d, v_1 and v_2, the corresponding value of x can be found.

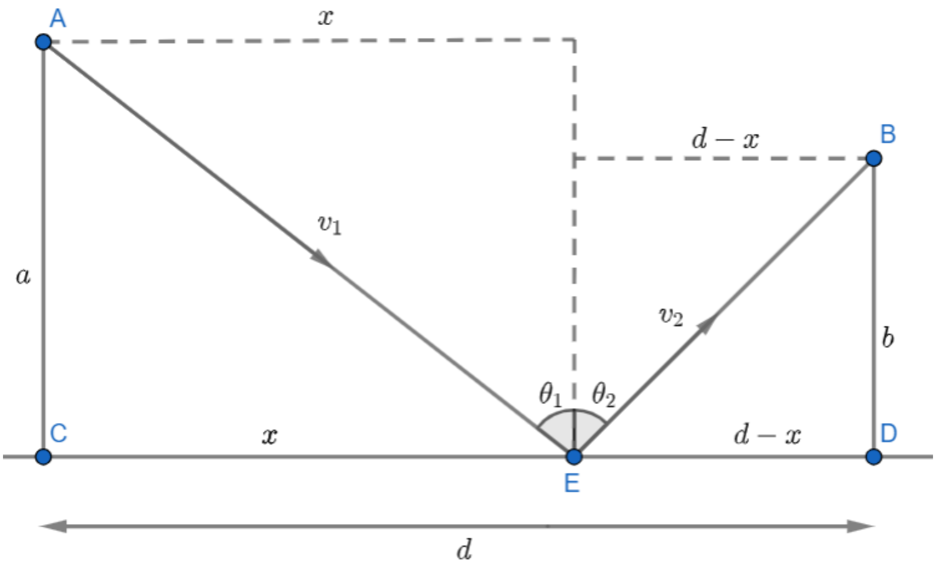
But why should we stop here? Manipulating this equation carefully produces something wonderfully surprising, which I will showcase now.

A sense of familiarity?

Rather than considering the distance along the river that the farmer should travel, there is no reason not to consider the angle at which they should approach and leave the river at. After all, the farmer will be travelling in a straight line. Now, let's consider the two angles θ_1 and θ_2 that are detailed below.



We can add in some line segments and lengths to help with determining the value of these two angles.



From the above diagram, we can produce the following equations for the angles θ_1 and θ_2:

$$\sin\theta_1 = \frac{x}{\sqrt{a^2+x^2}}$$

$$\sin\theta_2 = \frac{d-x}{\sqrt{b^2+(d-x)^2}}$$

Now let's revisit the equation we uncovered in the previous section.

$$\frac{x}{v_1\sqrt{a^2+x^2}} = \frac{d-x}{v_2\sqrt{b^2+(d-x)^2}}$$

This can be rewritten to:

$$\frac{1}{v_1} \times \frac{x}{\sqrt{a^2+x^2}} = \frac{1}{v_2} \times \frac{d-x}{\sqrt{b^2+(d-x)^2}}$$

A quick glance back at the expressions for the sine of the angles θ_1 and θ_2 reveals a beautiful simplification!

$$\frac{1}{v_1}\sin\theta_1 = \frac{1}{v_2}\sin\theta_2$$

The simplicity of this new equation is far more palatable than the one involving the distances shown previously. It is stunning how different they look when, under the hood, they are both the same equation.

But wait, there's more. Anybody that has studied the physics of waves will have seen this before. It is a rearrangement of Snell's law, which describes how waves behave as they pass from one medium to another.

If you cannot recognise the formula immediately, try multiplying both sides by the speed of light in a vacuum, c:

$$\frac{c}{v_1}\sin\theta_1 = \frac{c}{v_2}\sin\theta_2$$

The ratio of the speed of light in a vacuum to its speed (c) in a medium (v_1 or v_2) is known as the refractive index of the medium and is denoted with the letter n. This allows us to rewrite the above to the more recognisable version of Snell's Law:

$$n_1\sin\theta_1 = n_2\sin\theta_2$$

Whether or not you have fond memories of school physics lessons should not alter how surprisingly intriguing the appearance of Snell's law in this context is. But what is Snell's law and why does it have any right to dictate the motion of our farmer?

The link to light and refraction

When light (or any wave) passes over the boundary between two different materials, or mediums, it changes speed. This causes the wave to bend; this phenomenon is known as refraction. The angle that the incoming light ray makes with the normal to the boundary is called the angle of incidence or θ_1, and the angle that the ray makes with the normal as it leaves the boundary is called the angle of refraction or θ_2. This is shown on the next page.

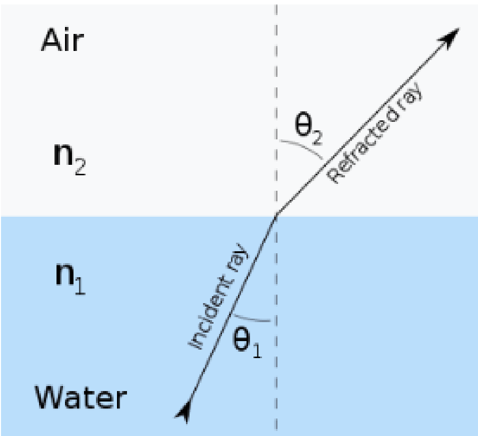


Image credit: https://en.wikipedia.org/wiki/Angle_of_incidence_(optics)

Every medium has an associated refractive index (n) that is equal to the ratio of the speed of light in a vacuum (c) to the speed at which light travels in the medium (v):

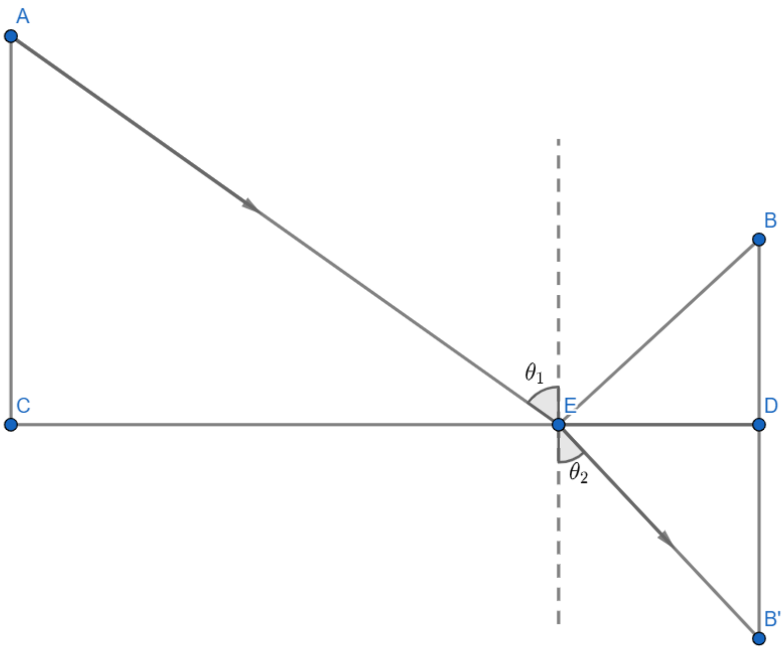
$$n = \frac{c}{v}$$

A refractive index closer to 1 means that light will travel faster in the medium; usually, the refractive index cannot be less than 1 because light cannot travel faster in a medium than it can in a vacuum.

Snell's law, which was stumbled upon earlier, quantifies the relationship between the angle of incidence (θ_1), the angle of refraction (θ_2) and the refractive indices of the two materials (n_1 and n_2):

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

We can see why Snell's law shows up in this problem by reflecting the point B about the river:



Hopefully, you can see that the path from A to B' looks eerily similar to the path of a light ray travelling from one medium to another. In the diagram, $\theta_1 > \theta_2$. This means that, if this were a light ray, it would be travelling from a medium with a lower refractive index into a medium with a higher refractive index, causing it to slow down. This is akin to what the farmer does: they slow down after filling up their bucket with water.

Furthermore, light follows Fermat's Principle. This, also known as the principle of least time, states that the path taken by a ray of light (or other wave) between two points is the path that can be travelled there in the least amount of time. This explains exactly why Snell's law appears here! The farmer trying to attend to his injured cow wished to, just like a ray of light, travel from one point to another *in the shortest amount of time.* This is why both objects will approach the river, or boundary between mediums, at the same angle and take the same path.

Conclusion

I believe that this surprising link between a farmer and the behaviour of light itself is the perfect end to this adventure.

From a tedious first approach, to stunning geometry and an unexpected application to physics, this problem illustrates wonderfully the joy of investigating a seemingly unimportant problem. It highlights the value of thinking about things more deeply and not just moving on once you have found a single working solution. I hope that this essay has encouraged you to go and find a problem to investigate for yourself.

But - if there is one thing to take away from all of this - it is that, in times of agricultural crisis, interesting mathematics has never been more important.

References:

https://www.youtube.com/watch?v=0SI3_XbItHo

https://www.youtube.com/watch?v=aW7R1U6FMLs&t=3s

https://en.wikipedia.org/wiki/Snell%27s_law

https://en.wikipedia.org/wiki/Angle_of_incidence_(optics)

https://en.wikipedia.org/wiki/Fermat%27s_principle

https://en.wikipedia.org/wiki/Refractive_index

The various diagrams were made by me on https://www.geogebra.org/geometry?lang=en

The graphs shown were also made by me on https://www.desmos.com/calculator/ravj6t8ovy