Unveiling the Mathematics and Applications of Fractals

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March 29, 2024

1 Introduction

The term "fractal" was coined by the mathematician Benoit Mandelbrot in 1975, deriving from the Latin word "fractus," which means broken or fragmented. Essentially, fractals are fascinating mathematical objects that exhibit self-similarity at different scales, meaning that as we zoom into a fractal, we discover smaller copies of the whole structure repeated infinitely within itself. [1]

This essay aims to explore the mathematics of fractals work and their applications in the real world.

2 Fractal Dimensions

Dimensions is a branch of mathematics used to describe the arrangement of objects. For instance, in Euclidean geometry (branch of mathematics that originated from the works of the ancient Greek mathematician Euclid, who lived around 300 BCE.), dimensions refer to the number of coordinates needed to specify the location of a point within a space.

A one-dimensional space, such as a line, requires only one coordinate (e.g., distance along the line) to uniquely identify any point within it. Similarly, a two-dimensional space, such as a plane, requires two coordinates (e.g., x and y coordinates) to locate any point within it. A three-dimensional space, such as our physical universe, requires three coordinates (e.g., x, y, and z coordinates)1.

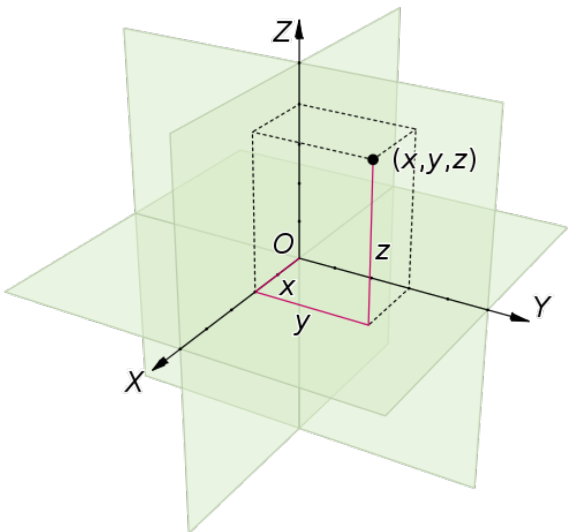

Figure 1: A point in three-dimensional Euclidean space can be located by three coordinates [2]

Euclidean dimensions are integer-valued and correspond to length, width, and height. However, the concept of dimension becomes more nuanced when we explore fractals as unlike Euclidean dimensions, fractal dimensions can be fractional or non-integer, reflecting the intricate and self-similar nature of fractal geometry.

The fractal dimension measures the complexity or "roughness" of a fractal pattern, describing its degree of self-similarity on different scales. Let's consider the example of the Sierpiński Triangle2, one of the most iconic examples of a fractals named after the Polish mathematician Wacław Sierpiński.

2.1 Sierpiński Triangle

Figure 2: Evolution of the Sierpiński Triangle[3]

2.1.1 Iteration

Iteration is one of the key mathematical concepts underlying fractals is. This is where a basic geometric shape or set of rules is repeated recursively to generate increasingly complex structures. (Recursive means that the process or pattern repeats itself in a self-similar manner, with each iteration building upon the results of the previous one. In other words, the the output of one iteration serves as the input for the next iteration.)

The Sierpiński Triangle starts with an equilateral triangle. At each iteration, the triangle is subdivided into four smaller triangles, and the central triangle is removed. This process repeats recursively, generating smaller copies of the original triangle within itself at each iteration. As we zoom into the triangle, we encounter self-similar patterns on different scales, which is a property of self-similarity is a defining characteristic of fractals.

2.2 Computing the Fractal Dimension

To compute the fractal dimension of the Sierpinski Triangle, we can use a mathematical technique known as box-counting[4].

Perhaps a more intuitive and simplified way to think about dimensions[5] is to consider these four shapes: a line, a square, a cube, and a Sierpiński Triangle. All of these shapes are self-similar, but they are not all fractals. Now imagine these shapes are made of some sort of metal which can allow us to compare how the mass changes of the object changes as we scale then as illustrated in the table below:

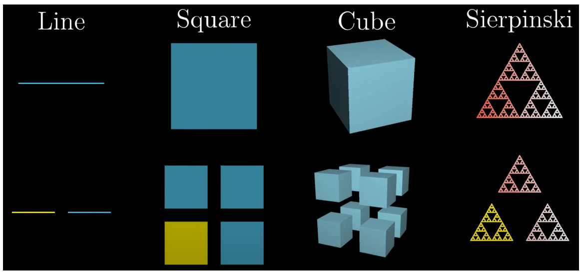

Figure 3: [5]

Shape	Scale Factor	Mass Scale factor
Line	$\frac{1}{2}$	$(\frac{1}{2})^1$
Square	$\frac{1}{2}$	$(\frac{1}{2})^2$
Cube	$\frac{1}{2}$	$(\frac{1}{2})^3$
Sierpiński Triangle	$\frac{1}{2}$	$\frac{1}{3}$

Starting with the line, if we were to scale the line by a factor of a $\frac{1}{2}$, then its mass will also be scaled down by $\frac{1}{2}$ as it would take two copies of the smaller to form the original shape. Likewise with the square and the cube, if we scaled the shapes by $\frac{1}{2}$, then the mass would be scaled by $\frac{1}{4}$ and $\frac{1}{8}$ respectively as shown in the table above.

Now, if we were to scale the Sierpiński Triangle $\frac{1}{2}$ down by a factor of $\frac{1}{2}$, its mass will decrease by a factor of $\frac{1}{3}$ as it would take three copies of that scaled down triangle to form the original triangle.

If we notice that for the line, the square and the cube, the factor by which the mass changed when we scaled the shape were integer powers of $\frac{1}{2}$ or $(\frac{1}{2})^D$ where D happens to be the dimension of that shape. Using this as our understanding of dimension, we can derive that the dimension for the Sierpiński Triangle must be some value D for which when we scale down our shape by a factor of $\frac{1}{2}$, its mass decreases by a factor of $(\frac{1}{2})^D$. And because our shape is self-similar, we know that this mass will be equal to $\frac{1}{3}$. This leads us to the equation:

$$M = \left(\frac{1}{2}\right)^D = \left(\frac{1}{3}\right)$$

The equation is the same as answering the question, "What power do I raise the number 2 to get 3?" and after taking logs we get:

$$2^D = 3$$

$$\log_2 3 \approx 1.585$$

Therefore, for the Sierpiński Triangle, the fractal dimension is ≈ 1.585.

2.2.1 Box-counting

Box-counting is a method used to estimate the fractal dimension of a pattern or set. It works by covering the pattern with a grid of squares of varying sizes, then counting the number of squares required to cover the pattern at each scale. By analysing how the number of squares changes with the size of the squares, we can estimate the fractal dimension of the pattern. We can use this method is because fractals patterns consist of smaller parts of the pattern that resemble the whole pattern.

Below is the general formula to compute the dimension D of a fractal object with a grid of boxes using the box counting method:

$$D = \lim_{\epsilon \to 0} \frac{\log(N(\epsilon))}{\log(1/\epsilon)}$$

where:

- $N(\epsilon)$ is the number of boxes of size ϵ required to cover the fractal object.
- ϵ is the scale of the box (or the resolution)

3 Example Fractals: The Mandelbrot Set

The Mandelbrot set probably one of the most famous and visually stunning fractals which was discovered by Benoit Mandelbrot in the 1970s. But the thing makes the Mandelbrot set so special is its remarkable properties related to stable and unstable iterations.

3.1 Brief Overview of Complex Numbers

To understand the Mandelbrot Set, we must first familiarise ourselves with the complex numbers.

Complex numbers in mathematics are numbers that consist of a real part and an imaginary part and are represented in the form of:

$$a + bi$$

where a is the real part, b is the 'imaginary part', and i is the imaginary unit $\sqrt{-1}$. The real part is plotted along the x-axis and the imaginary part along the y-axis.

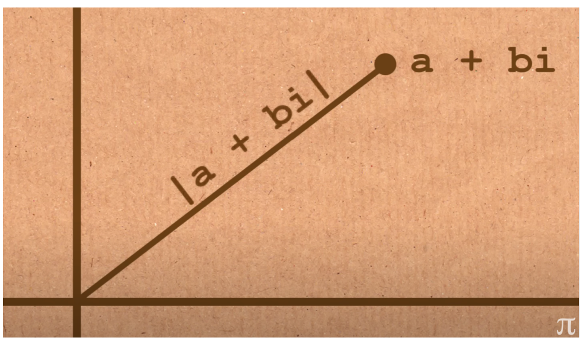

Figure 4: Representing Complex Numbers[6]

Complex numbers are different from real numbers. Real numbers (which are a subset of the complex numbers) are numbers that can be found on the number line and include integers (whole numbers), rational numbers (fractions), and irrational numbers (numbers that cannot be represented as a fraction) such as π and $\sqrt{2}$, as shown in the Venn Diagram in 5

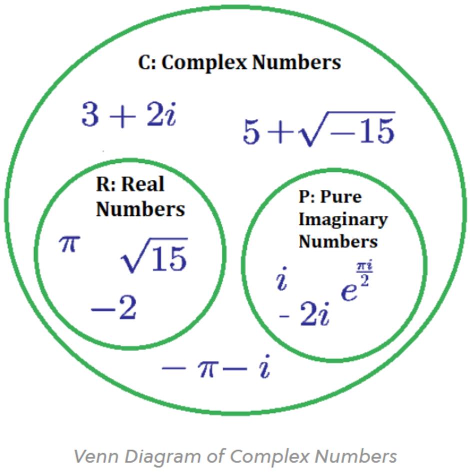

Figure 5: Complex Numbers from https://brilliant.org/wiki/complex-numbers/

The Mandelbrot set is generated using an iterative process involving complex numbers and the complexity of the Mandelbrot set arises from the interaction between different regions of the complex plane.

The iteration used is:

$$z_{n+1} = z_n^2 + c$$

where z_n is a complex number at iteration n and z_{n+1} is the next iteration, and c is the constant complex number which acts as a parameter of the equation. We start with an initial value for z_0 and repeat the equation to generate subsequent values.

The Mandelbrot set is represented in the complex plane because each point in the complex plane corresponds to a unique complex number c, allowing us to represent the entire set of complex numbers in a two-dimensional space. We need to use the complex plane to model the Mandelbrot set because the iterative process involves squaring complex numbers and adding constant complex numbers, which leads to patterns and structures that cannot be modelled with real numbers alone.

3.2 Stable and Unstable Iterations

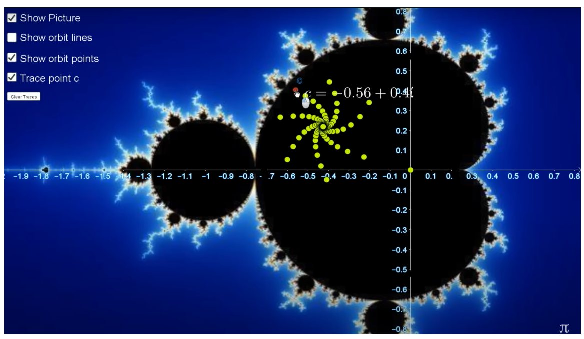

Figure 6: Mandelbrot Set [7]

To determine whether a complex number c belongs to the Mandelbrot set, we iterate the equation a finite number of times and observe the behaviour of the sequence generated by the iterations.

The boundary of the Mandelbrot set separates the stable region (inside the set) from the unstable region (outside the set). For points within the Mandelbrot set, the iterative process remains *stable*, meaning that the sequence of values generated by the iterations remains bounded and does not diverge to infinity. These points are said to *converge* under iteration.

Points outside the Mandelbrot set, on the other hand, have *unstable* behaviour under iteration. As the iterative process progresses, the sequence of values generated by the iterations may *diverge* to infinity, meaning that the magnitude of the complex numbers becomes arbitrarily large, and these points are said to *diverge* under iteration.6

3.2.1 Colours

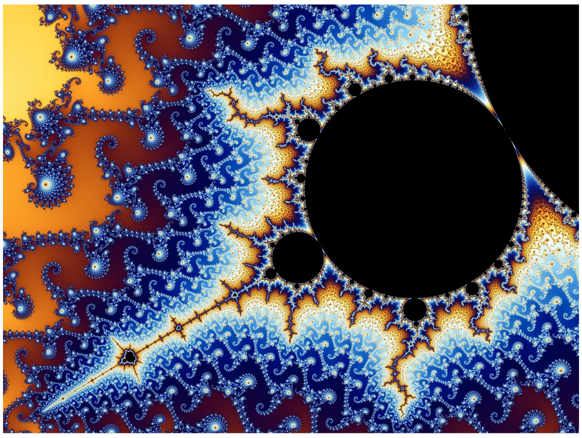

Figure 7: “Antenna” of the satellite.

The colours in the Mandelbrot Set, as shown in 7 are typically assigned based on the number of iterations it takes for each point in the complex plane to escape from a predefined boundary. The boundary is usually set at a certain magnitude, beyond which the iterative process is considered to diverge towards infinity.

In general, points that are inside the Mandelbrot set are typically coloured black or some other constant colour because these points are part of the set itself and are considered to be “stable” under iteration.

Points that are outside the Mandelbrot set, are assigned colours based on the number of iterations it takes for them to escape. This is often done using a colour gradient, where each colour represents a different number of iterations. For instance, points that escape quickly, within a small number of iterations, might be assigned a colour closer to the beginning of the gradient (e.g. blue), while points that take longer to escape might be assigned colours closer to the end of the gradient (e.g. red).7

4 Applications of Fractals

Fractals aren’t just for being pretty, they have real uses too!

Fractal geometry is used to model natural phenomena characterised by irregularity and self-similarity, such as coastlines, clouds, mountains, and biological structures. Fractal algorithms enable scientists to create realistic simulations of natural processes, aiding in understanding and predicting erosion, fluid dynamics, and growth patterns in plants and animals.

In fact, it was Mandelbrot’s fascination with irregular shapes and patterns from nature that led him to realise that traditional Euclidean geometry, which deals with smooth and regular shapes, was inadequate for describing such intricate and fragmented forms.

Points near the boundary of the Mandelbrot set correspond to chaotic behaviour, where small changes in initial conditions lead to drastically different trajectories. While the term “chaos” might suggest disorder and randomness, chaos theory actually deals with deterministic systems—those governed by fixed rules—but ones that display unpredictable behaviour over time due to their inherent complexity. One of the key connections between fractals and chaos theory is chaotic attractors. An attractor is a set of states to which a dynamical system evolves over time, representing the system’s long-term behaviour. The famous “Butterfly effect,”[8] popularised by Edward Lorenz, is an example of a chaotic behaviour that illustrates the sensitivity of chaotic systems to initial conditions. It suggests that a small change in the initial state of a system can lead to vastly different outcomes over time, via the analogy of a tornado being influenced due to a butterfly flapping its wings several weeks earlier.

5 Concluding Reflections

Fractals, and in particular the likes of Mandelbrot set, have captivated mathematicians, scientists, and artists alike. As we zoom into different regions of the set, we encounter an endless array of intricate patterns, spirals, and filaments, revealing self-similar structures on increasingly smaller scales. This infinite richness of detail is what makes the Mandelbrot set a mathematical masterpiece. No matter how far we zoom in, there are always new features to discover, each more intricate than the last. The Mandelbrot set is teeming with connections to fundamental mathematical constants, such as π, and Fibonacci (ϕ). While the appearance of π and Fibonacci may not be immediately apparent in the Mandelbrot set's visual patterns, their presence emerges through deeper mathematical analysis and exploration. As we continue to unveil the world within fractals, we not only deepen our understanding of complex systems, but also appreciate the beauty and elegance of mathematical discovery and the practical applications of the same.

I hope that readers of this essay have been left with a newfound appreciation of mathematics and fractals that inspire curiosity, creativity, and wonder.

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