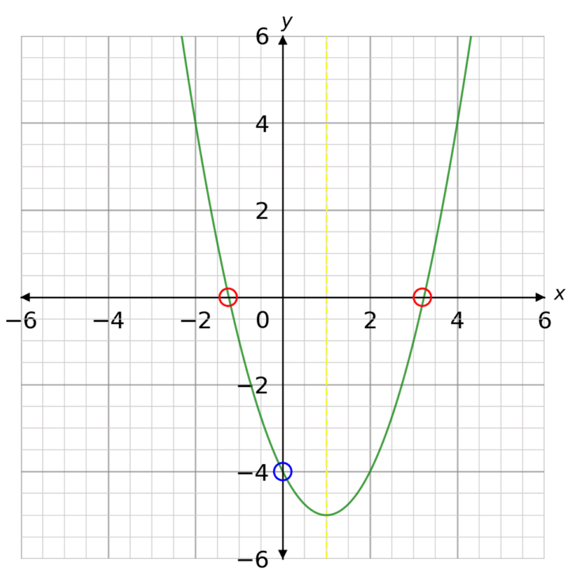
# Transcendental Numbers : a Short Introduction

## What is a Transcendental Number?

The word ‘transcendental’ is derived from the mediaeval latin words, ‘transcendere’ meaning to surmount and ‘scandere’ meaning across/beyond and is typically taken to describe that which relates to the spiritual realm. However, in the context of mathematics it has an entirely separate meaning. A quick google search will find the definition to be along the lines of “a number (possibly complex) that is non-algebraic” but this just begs the question what constitutes an algebraic number?

An algebraic number is a number that can be obtained as the root of an integer polynomial equation(a polynomial with only integer coefficients), take for example the number ‘2’, Which can be obtained from many different integer polynomials such as:
$2x^2 - 8 = 0$
or $x - 2 = 0$
Both of these polynomials if solved will give at least one solution equalling ‘2’, but why? What does the solution mean? To explain take $2x^2 - 8 = 0$ which actually is the first step of solving the equation $y = 2x^2 - 8$ as $y = 0$ at the points where where the graph crosses the $x$-axis.
Below is a visual representation of $y = 2x^2 - 8$ where it crosses the $x$-axis.




(“Quadratic Graphs Worksheets, Questions and Revision | MME”)

The circled points on the graph are showing what is known as the 'roots' of the graph or in the vernacular the points at which $y = 2x^2 - 8$ crosses the $x$-axis. An algebraic number is any number of which an integer polynomial can be constructed to give the desired number as a root.
As such find a possible integer polynomial for each of the following 3 numbers (possible answers are detailed below)

1. $\sqrt{2}$
2. $\frac{12}{25}$
3. $676$

Answers:

1. e.g. $x^2 - 2 = 0$ , $x^3 - 2\sqrt{2} = 0$
2. e.g. $25x - 12 = 0$ , $50x - 24 = 0$
3. e.g. $x^2 - 456976 = 0$

A transcendental number would therefore be any number that does not fit this description of an algebraic number.

## The History of Transcendental Numbers

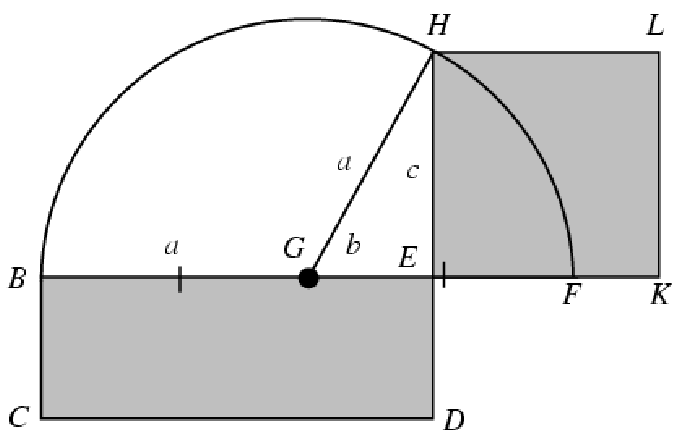
The existence of transcendental numbers was first proven in 1844 by Joseph Liouville but it was not until 7 years later in 1851 that he proved the transcendence of a specific case, now known as the Liouville constant, which takes the form $\sum_{n=1}^{\infty} 10^{-n!}$. A factorial (denoted by an exclamation mark '!') in this case is an integer which is the product of all prior integers e.g. 4! = $4 \times 3 \times 2 \times 1 = 24$ and when 10 has a negative exponent 'n' then it can also be displayed as 0.(n number of 0's)1. The final necessary piece of notation to understand is the $\sum_{n=1}^{\infty}$ the greek letter sigma ($\sum$) means the sum of, and the number at the top and bottom in this case, n=1 and ∞, mean that you sum all of the terms in the series starting with n=1 all the way up to infinity. For this example the sum of the first few terms would look as follows; $10^{-1} + 10^{-2} + 10^{-3} + 10^{-4} + (...) + 10^{-\infty} =$ 0.110001000000000000000001 (and so on).
Another way of thinking about this number is to imagine 0 with infinite .00000000s and replacing every ' n! ' th 0 with a 1 where $n \in Z$ (n is an integer) so the 1!=1, 2!=2, 3!=6, 4!=24, 5!=120,(...)th 0's are replaced with 1's.

Other much more common transcendental numbers include $e$ (Euler's number) and π (pi), $e$ was the first number proven to be transcendental without being constructed specifically for that purpose (unlike the Liouville constant) and this was proven by Charles Hermite in 1873 via a proof by contradiction. By further using this proof and Euler's identity ($e^{i\pi} = 1$ ), Carl Louis Ferdinand

Von Lindemann constructed a proof by contradiction to show that if $e$ is transcendental then π must also be.

## Applications of the Transcendentality of π

Around approximately 300 BC a Greek man by the name of Euclid released a series of 13 books titled 'The Elements' detailing almost all major proofs at the time. Book $I$ of 'The Elements' concerns plane Geometry, now often referred to as 'Euclidean Geometry'. Within this text Euclid discusses the squaring or 'quadrature' (construction of a square of equal area, using only a compass and an unmarked straight edge) of many different shapes, the first of which is the rectangle. The method for such a construction is as follows:




Dunham, William W. *Journey Through Genius*. Penguin, 1990. *Michigan State University Libraries*.

Let $BCDE$ be an arbitrary rectangle, extent the line $BE$ to point $F$ such that $ED = EF$
Next a semicircle must be constructed with $G$ as its centre such that $G$ is equidistant from $B$ and $F$.
Then extend the line $DE$ until it meets the semicircle giving the first side of the square then simply construct the square $EHLK$. □

It can now be said that $BCDE$ and $EHLK$ have the same area. Euclid then goes on to show methods for the quadrature of many other regular polygons, but not until Book $XII$ does Euclid show Hippocrates' method for the quadrature of the lune (a crescent moon shape) Which is the closest anyone up until that point had come to the quadrature of a circle. Right up to 1882 this remained the case, however Lindemann's proof of π's transcendentality meant that as π is not the solution to any integer polynomial, as previously mentioned, and therefore is not constructible (i.e. you cannot draw a line of length 'π') and as the area of a circle of radius 1 is given as π this would mean the square would have to have sides of length $\sqrt{\pi}$ which would be impossible.

## A More Recent Example

In Paris on the 8th of August, 1900 David Hilbert (one of my personal favourite mathematicians) , often known for his use of the 'Hilbert Hotel' to describe infinities, announced 10 of his soon-to-be published 23 problems, which he believed were the 23 most important problems in mathematics at the time. Many still remain so to this day, such as the 8th problem, the Riemann Hypothesis, which became one of the Clay Institute's 7 millennium problems with a bounty of $1Million for whomever can solve it. However, the one of Hilbert's problems that links to the topic of transcendence in numbers is the 7th which is as follows;

Is $a^b$ transcendental given that $a \neq 0, 1$ and algebraic and $b \notin R$ and is algebraic?

An example of such a number would be $2^{\sqrt{2}}$ which is dubbed the Gelfond-Schneider constant after the two men that discovered the solution in 1934 to Hilbert's problem, Aleksandr Gelfond and Theodor Schneider (however, it is also commonly referred to as the Hilbert Number for obvious reasons).

## Conclusion

Transcendental Numbers aren't a very common subject of discussion and most people (especially people who haven't studied mathematics at a higher level) haven't even heard of them. However they have a magical quality to them as both π and *e* appear an inordinate amount both in mathematics and in real life. For instance not only is π the ratio of the circumference of a circle to its diameter but it is also the average sinuosity of a river, making it completely understandable as to how they got dubbed transcendental.

Unfortunately the proofs for transcendental numbers are all too long and complex to be featured in this essay, but if you are intrigued I encourage you to watch The Mathologer YouTube video on the proof of Liouville's Number being transcendental as he explains it both beautifully and concisely.