Solving Greek Problems: looking into field theory and origami

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Introduction:

Greek problem is a set of geometries problems asked by ancient Greek mathematicians. This essay mainly explores some Greek problems: including regular n-sided polygon construction ,angle trisection and more and solves them through field theory and origami. Historic events behind the questions and furthermore about field theory and origami in mathematics would be also discussed.

1.1 Classical constructions:

Ancient Greek mathematicians mainly focuses on patterns, structures , space, apparent change and measurement, which now knowns as the study of Euclidean geometry. In the time, straightedge and compass constructions (classical constructions) are favored by Greek mathematicians .This method of geometric construction relies on the use of only two basic tools: a straightedge, a ruler without markings, and a compass, a circular drawing instrument. Greek mathematicians believed that a straight line and the circle are the most fundamental shapes in nature, with these seemingly simple tools, they believed everything could be constructed.

The allure of straightedge and compass constructions lies in their elegance and simplicity. With just the ability to draw straight lines and circles, complex geometric shapes could be creating intricate design and be constructed by individuals. This minimalistic approach to construction embodies the spirit of mathematical purity, emphasizing the power of logic and reasoning over other tools or measurements.

1.2 Greek Problems:

Fascinated by the pursuit of geometric perfection, ancient Greeks had never stop exploring it. Although mathematicians such as Euclid and Archimedes developed a comprehensive system of geometry, which included principles for constructing a wide range of geometric objects using only a straightedge and compass, but there are problems could not be solved in the days. Famous problems like circle squaring, cube duplication and angle trisection are asked to be solved by classical constructions which is now proven to be impossible after more than 2000 years they are formulated.

Today, we would explore the regular n-sided polygon construction problem solved by Carl Friedich Gauss and the Circle squaring, cube duplication and angle trisection problem by field theory to be insoluble.

2.1 Regular Heptadecagon construction problem:

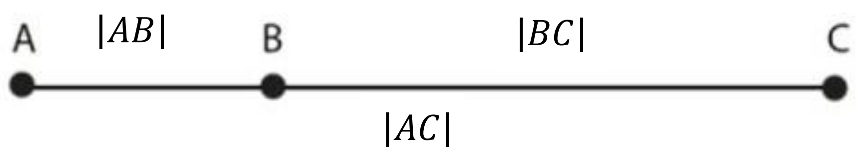
Regular Heptadecagon (17-sided polygon) construction problem is a branch of the regular n-sided polygon construction problem. Solving it was one of the most outstanding achievements that Gauss had made.

In 1796, at the University of Gottingen, young mathematician Carl Friedich Gauss received a homework assignment from his professor as a student. He solved it overnight. When he handed in his work to his teacher his teacher could not believe it. The homework assignment with the problem Regular Heptadecagon construction was a topic that professor was researching but stuck himself but accidentally given to Gauss. This problem had troubled mathematicians for ages. But Gauss did, he solved it, genially.

2.2 Numeric expressions of classical constructions

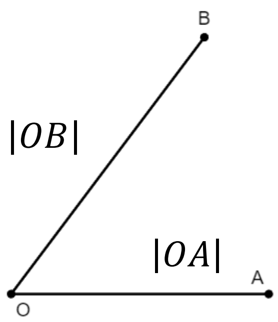
To understand Gauss's proof, basic knowledge about numeric expressions of classical constructions is required. Numeric expressions of classical constructions means that as a unit length one is given, others could be constructed by straightedge and compass. Today, only addition, deduction, multiplication, division and square root would be introduced and required for Gauss's proof.

Addition and deduction are simple to do. Just simply extended or deduce the length of a line segment.



Addition: $|AB| + |BC| = |AC|$ Deduction: $|AC| - |BC| = |AC|$

Multiplication and division are not as straightforward. Properties of similar triangle has to be considered to do.



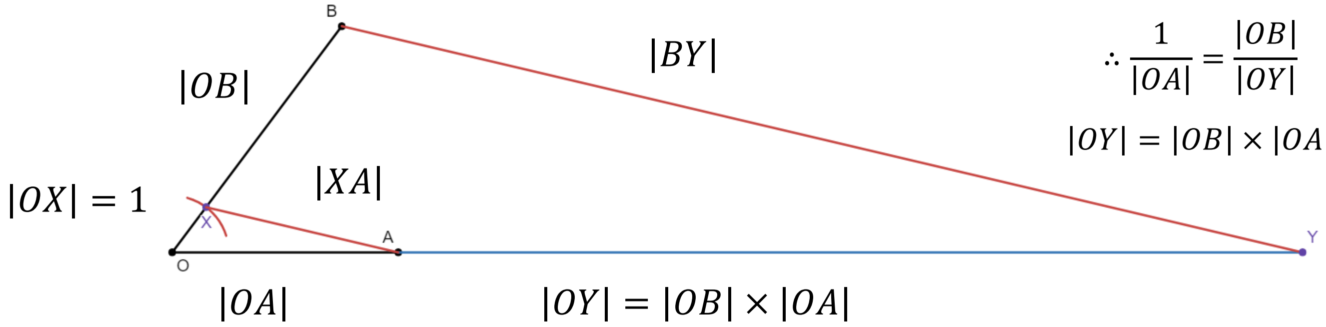
By given length $|OA|$,$|OB|$ and unit length 1, place $|OA|$,$|OB|$ at any angle. Swing an arc with unit length 1 to $\overline{OB}$. Label interception X. Connect $\overline{AX}$ and construct line $\overline{BY}//\overline{AX}$. The length $\overline{OY}$ is the product of $|OA|$,$|OB|$. This could be proven by noticing $\Delta OAX \sim \Delta OBY$.

Multiplication:

$$\Delta OAX \sim \Delta OBY$$

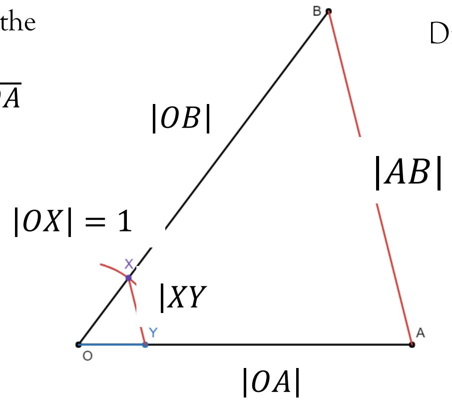
$$\therefore \frac{1}{|OA|} = \frac{|OB|}{|OY|}$$

$$|OY| = |OB| \times |OA|$$



To form division, we also construct point X on the line $\overline{OB}$ with $|OX| = 1$ which is the same as multiplication. Connect $\overline{BA}$ and extend X to $\overline{OA}$ and form $\overline{XY}$ which $|XY| // |BA|$. The length $|OY|$ is $\frac{|OA|}{|OB|}$. The proof is also simple. By noticing $\Delta OXY \sim \Delta OAB$. Square root in geometry could be also done easily.

$$|OY| = \frac{|OA|}{|OB|}$$

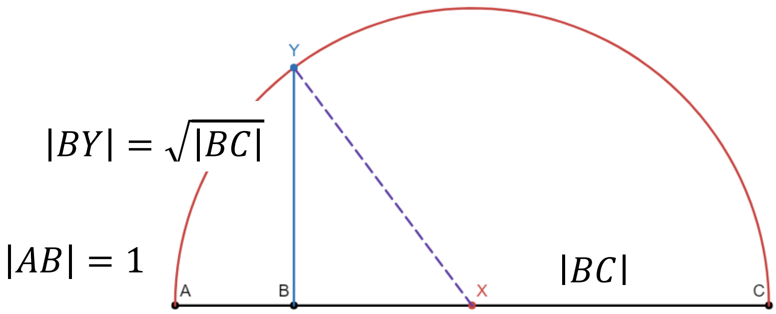


Division:

$$\Delta OXY \sim \Delta OAB$$

$$\therefore \frac{1}{|OY|} = \frac{|OB|}{|OA|}$$

$$|OY| = \frac{|OA|}{|OB|}$$



Square root:

$$|AX| = |XC| = |XY| = r$$

$$|AC| = 2r = |BC| + 1$$

$$|Bx|^2 + |BY|^2 = |XY|^2$$

$$(r-1)^2 + |BY|^2 = r^2$$

$$|BY|^2 = 2r - 1$$

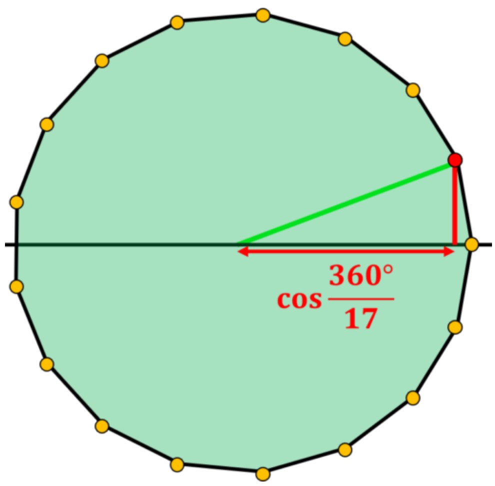
$$|BY|^2 = |BC|$$

$$|BY| = \sqrt{|BC|}$$

By given length $|BC|$ and unit length 1 $|AB|$, find mid-pt X and construct a semicircle with diameter $\overline{AC}$ .Draw line $\overline{BY} \perp \overline{AC}$ such that pt Y lies on the semicircle. We get $|BY| = \sqrt{|BC|}$. By properties of a circle, we notice that $|AX| = |XC| = |XY|$ which is the radius. Then we can proof:

2.3 Finding $\boldsymbol{cos}\left(\frac{2\pi}{17}\right)$

Heptadecagon



After understanding some numeric expressions of classical constructions, we can now look more into how Gauss constructed a heptadecagon. A regular heptadecagon's all side have same length with vertex all lies on the same circle. By these properties, finding the length of the side would be the only requirement of constructing a heptadecagon. To find the side length, Gauss chosen to find $cos\left(\frac{2\pi}{17}\right)$. Since the angle from the center to any vertex is evenly divided into 17 equal parts.

To find $cos\left(\frac{2\pi}{17}\right)$, we consider the double-angle formula and supplementary-angle formula:

$$cos\left(\frac{2n\pi}{17}\right) = 2\,cos^2\left(\frac{n\pi}{17}\right) - 1$$

$$cos\left(\frac{16\pi}{17}\right) = cos\left(\pi - \frac{\pi}{17}\right) = -\cos\left(\frac{\pi}{17}\right) = -x$$

Therefore,

$$-x = -\,cos\left(\frac{\pi}{17}\right) = cos\left(\frac{16\pi}{17}\right) = 2\cos^2\left(\frac{8\pi}{17}\right) - 1 = 2\left(2\cos^2\left(\frac{4\pi}{17}\right) - 1\right)^2 - 1$$

$$cos\left(\frac{4\pi}{17}\right) = 2\,cos^2\left(\frac{2\pi}{17}\right) - 1 = 2\left(2\cos^2\left(\frac{\pi}{17}\right) - 1\right)^2 - 1 = 2(2x^2-1)^2 - 1$$

By substituting and simplifying,

$$-x = 2\left(2\cos^2\left(\frac{4\pi}{17}\right) - 1\right)^2 - 1 = 2(2[2(2x^2-1)^2-1]^2-1)^2 - 1$$

$$32761x^{16} - 131072x^{14} + 212992x^{12} - 180224x^{10} + 84480x^8 - 21504x^6 + 2688x^4 - 128x^2 + x + 1 = 0$$

Through solving x,

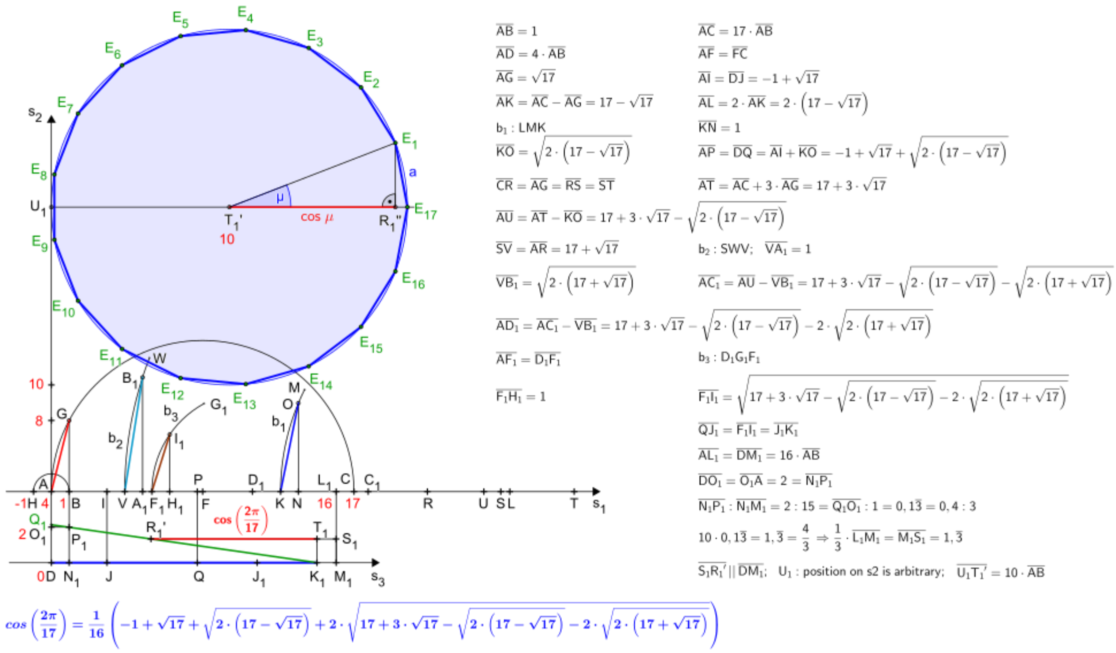
$$x = cos\left(\frac{\pi}{17}\right) = \frac{\sqrt{34-2\sqrt{17}} - \sqrt{17} + 1 + 2\sqrt{\sqrt{34-2\sqrt{17}} + \sqrt{17} - 1}\sqrt{\sqrt{17+4\sqrt{17}}}}{16}$$

By double angle formula,

$$cos\left(\frac{2\pi}{17}\right) = \frac{\sqrt{17} - 1 + \sqrt{34-2\sqrt{17}}}{16} + \frac{\sqrt{17 + 3\sqrt{17} - \sqrt{34-2\sqrt{17}} - 2\sqrt{34+2\sqrt{17}}}}{8}$$

2.4 Constructing a Heptadecagon

After finding $cos\left(\frac{2\pi}{17}\right)$, it can be investigated that the form Gauss founded is in terms of addition, deduction, multiplication, division and square roots which could be constructed classically shown in 2.1.2. The construction:



Solving the problem of regular heptadecagon construction make Gauss famous in mathematics. Gauss then focused on solving the problem on the regular n-sided polygon construction problem.

2.5 Regular n-sided polygon construction problem

For regular n-sided polygons, not all of them could be constructed through straightedge and compass. In 1801, Gauss published a book Disquistiones Arithmeticae, he was only 24 when he first published it. The first chapter in the book, in not just include his solution for constructing a heptadecagon, he also formulates s sufficient condition for the constructability of regular polygons. The formula suggests when n satisfy :

$$n = 2^k \prod F_i \quad k = 0,1,2,3 \cdots$$

If n is the product of a power of 2 and any number of distinct Fermat primes (including none), the regular n-gon could be constructed with straightedge and compass. Where Fermat primes is a prime number which is possible to be written in the following form:

$$2^{(2^i)} + 1$$

The only known Fermat primes till 2023 are 3, 5, 17, 257, and 65537. By using this result and applying Gauss's formula we could try find constructable Ns. F or example,

$$17 = 2^0 \cdot \left[2^{(2^2)} + 1\right]$$

Since 17 could be written in this form, it could be classically constructed as done in 2.4. Although Gauss stated without proof that this condition was also necessary, but never published his proof. Full proof of necessity was given by Pierre Wantzel in 1837. The result is known as the Gauss–Wantzel theorem.

By this, the regular n-sided construction problem is totally solved. Now we are moving on to other famous Greek problems: circle squaring, cube duplication and angle trisection.

3.1 Circle squaring, cube duplication and angle trisection

Circle squaring, cube duplication and angle trisection are really famous Greek problems asked by the Greeks but proved after 2000 years later which they are not constructable. To understand how they are solved, we have to know what these three problems are asking about.

The Circle squaring problem is asked to construct a square with equal area as a given circle.

The cube duplication problem is asked to construct a cube twice the volume as the given cube.

The angle trisection problem is asked to construct a given angle into three equal parts.

These three famous are all asked to be constructed by straightedge and compass.

3.2 Field theory and constructable number

Field theory is a branch of abstract algebra in mathematics which is closely related to group theory, it is also a fundamental tool for solving the Greek problems mentioned.

Field theory was first invented by two young mathematicians Niels Henrik Abel and Evariste Galois. A field is a set on which addition, subtraction, multiplication and division are defined and behave as the corresponding operations on rational and real numbers. By using the idea of field theory, Galois abled to proof that the quintic equation does not exist. Also, solving the Greek problems.

If and only if a number could be straightedge and compass constructed in finite steps with give unit length 1, it is defined as a constructible number. A constructable number are all in forms as sums, remainders, products, divisors and square roots of integers. For instance, $cos\left(\frac{2\pi}{17}\right)$ is in the group (field) of constructable number. But to distinguish a constructable number clearly and strictly, mathematicians formed this condition:

When a number k is a constructable number, k must be a root of a irreducible polynomial

$$a_n x^n + a_{n-2}x^{n-2} + a_{n-3}x^{n-3} \cdots + a_1 x + a_0 = 0$$

where,

$$a_i \in Q \text{ and } n = 2^m \qquad m = 0,1,2,3 \cdots$$

By using this condition, we can see that $cos\left(\frac{2\pi}{17}\right)$ is a constructable number. Since $cos\left(\frac{2\pi}{17}\right)$ is a root of the following while the highest power 16 is 2^4.

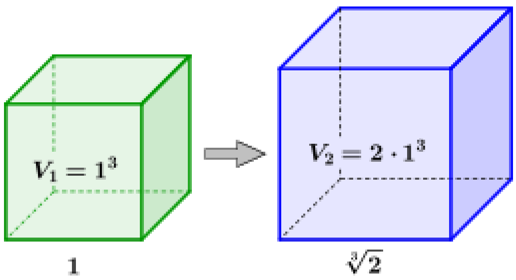
$$32761x^{16} - 131072x^{14} + 212992x^{12} - 180224x^{10} + 84480x^8 - 21504x^6 + 2688x^4 - 128x^2 + x + 1 = 0$$

This could be used to solve the three famous Greek problems too.

3.3 Solving Greek Problems

To solve the Greek problems, we have to identify what number are needed to construct and is the number a constructable number.

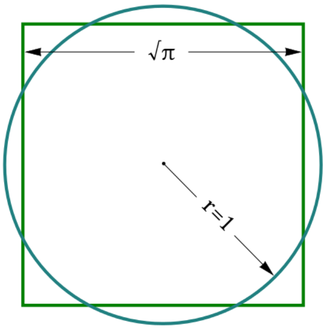
For the cube duplication problem, we could first assume the given cube have sides with unit length 1 for simplicity. Since we have to construct a cube twice as the volume as given, the side lengths of the cube have to be $\sqrt[3]{2}$. So, is $\sqrt[3]{2}$ a constructable number would be the question to solve.



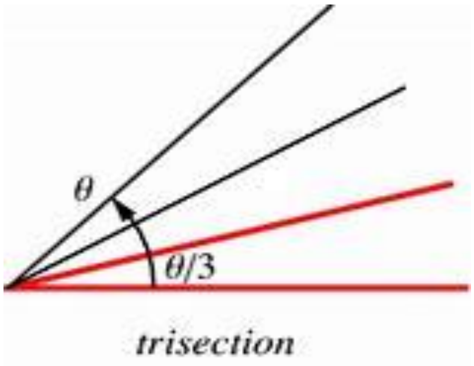
We can represent $\sqrt[3]{2}$ as a root of an irreducible polynomial,

$$x^3 - 2 = 0$$

Since the highest power in this polynomial is 3 which is not a power of 2, $\sqrt[3]{2}$ is not constructable. Through this, the cube duplication problem is solved.



By using a similar way, we can also solve the circle squaring problem. We assume the given circle have a radius one and area π, to form a square with same area, it's side would be $\sqrt{\pi}$. But since π is a transcendental number which is not the root of a non-zero polynomial with zero degree with rational coefficients, π and $\sqrt{\pi}$ are not constructable. For π, it can be constructed as the circumference of a circle but not a straight line, and the circle squaring problem though not possible to solve but exist lots of approximations like by taking π to 22/7.



The angle trisection would be a bit more complex to solve. We are given the angle θ and could get length $cos\,\theta$ easily with a similar presider as part 2.3. To trisect θ with known $cos\,\theta$, we have to find length $cos\left(\frac{\theta}{3}\right)$. To determine if $cos\left(\frac{\theta}{3}\right)$ is constructable, we can consider the tribble angle formula.

$$cos\,\theta = 4\,cos^3\left(\frac{\theta}{3}\right) - 3\,cos\left(\frac{\theta}{3}\right)$$

With $cos\,\theta$ as a constant and $cos\left(\frac{\theta}{3}\right)$ as an unknown. For simplicity we set $cos\left(\frac{\theta}{3}\right)$ as x. Through arrangements,

$$4x^3 - 3x - cos\,\theta = 0$$

By investigating the about equation, we can see that in general cases $cos\left(\frac{\theta}{3}\right)$ or x is not constructable since the highest polynomial is 3. But in special cases, for example when $\theta = 90°$

$$4x^3 - 3x - 0 = 0$$

$$4x^2 - 3 = 0$$

It could be seen that the polynomial is reduced into highest polynomial as 2. Not only when $\theta = 90°$, cases like $\theta = 45°, 180°, 360°\ etc$ could be constructed by straightedge and compass. For the angle trisection problem, despite special angles, mainly it could not be trisected. Now, all three Greek problems are solved. Now we could look into origami and see how origami relates to Greek problems.

4.1 Origami

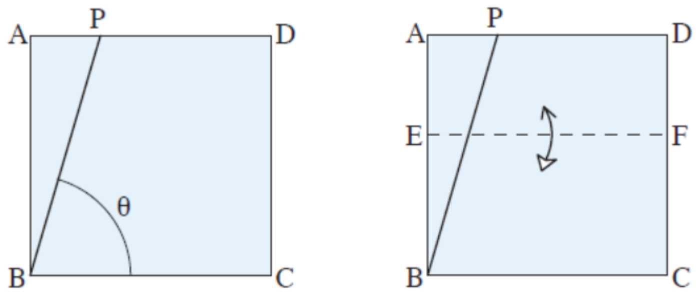
Origami comes from the words 'ori' (to fold) and 'kami' (paper). It first appears in Japan which Japanese monks used paper for ceremonial purposes around 600A.D. Modernly, origami is now viewed as a puzzle follows a sequence of folds to get shapes. It can be also viewed as a new type of geometry construction way as elegant and simple as straightedge and compass construction.

In the following, origami would be used to solve Greek problems instead of classical constructions.

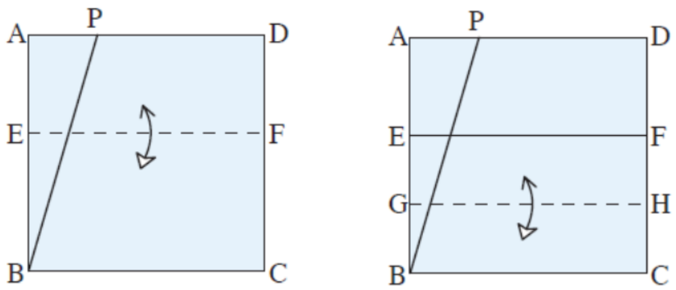
4.2 Solving angle trisection

Although we have shown that using only compass and straightedge to divide angle θ into equal thirds in general is undoable. Through origami, we could trisect any acute angle.

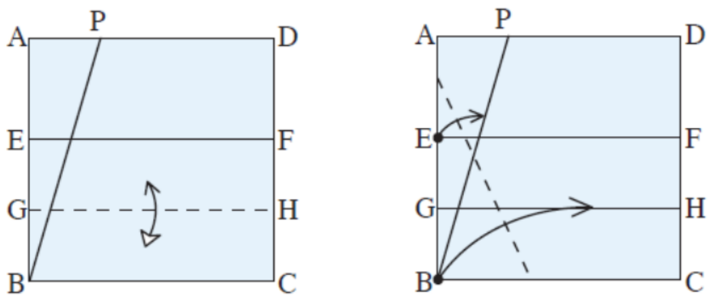
To trisect the angle PBC, we first fold any parallel line to BC,



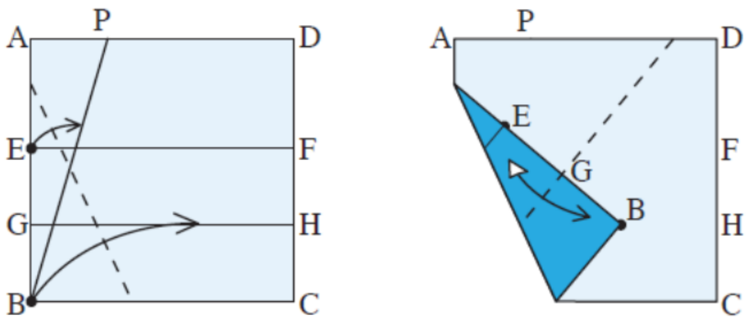
Then we fold a line from BC to EF such that the line lies in the middle between them.



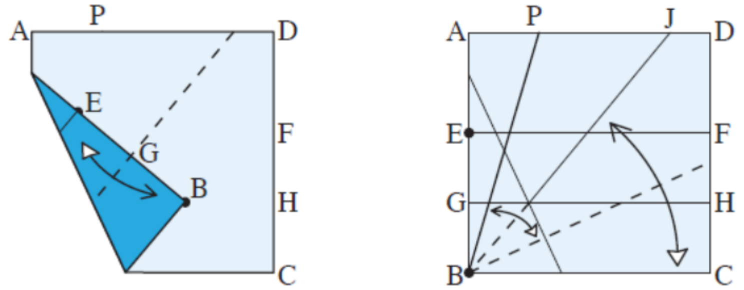
Next, we fold pt B to line GH and pt E to line PB.



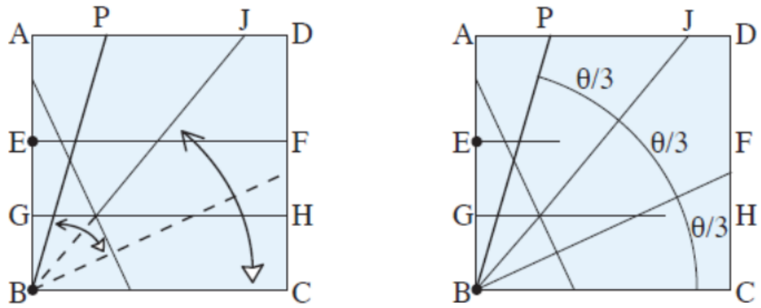
Afterwards, create a new line by folding B to E, then unfold.



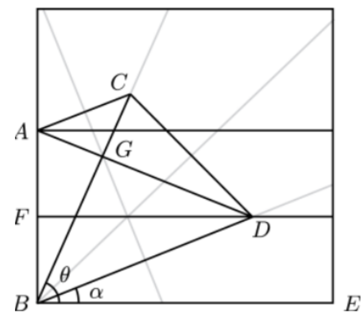
Extended line BC to BJ to form a new line, next unfold.



Finally, the angle is trisected.



To proof how it work,



Because BE and DF are parallel.$\angle DBE = \angle BDF$, Because DF is the altitude of the isosceles triangle ABD , $\angle BDF = \angle ADF$. Thus $\alpha = \angle DBE = \angle BDF = \angle ADF$. Now, $ABDC$ is an isosceles trapezoid and ABD is an isosceles triangle, so ABD and BCD are congruent isosceles triangles. Thus $\angle CBD = \angle ADB = \angle BDF + \angle ADF$.

It follows that $\theta = \angle CBE = \angle DBE + \angle CBD = \angle DBE + \angle BDF + \angle ADF = 3\alpha$.

Through origami, we can trisect any acute angle which classical construction cannot. Now we can investigate how to fold a regular n- sided polygon through origami.

4.3 Solving the regular n- sided polygon problem

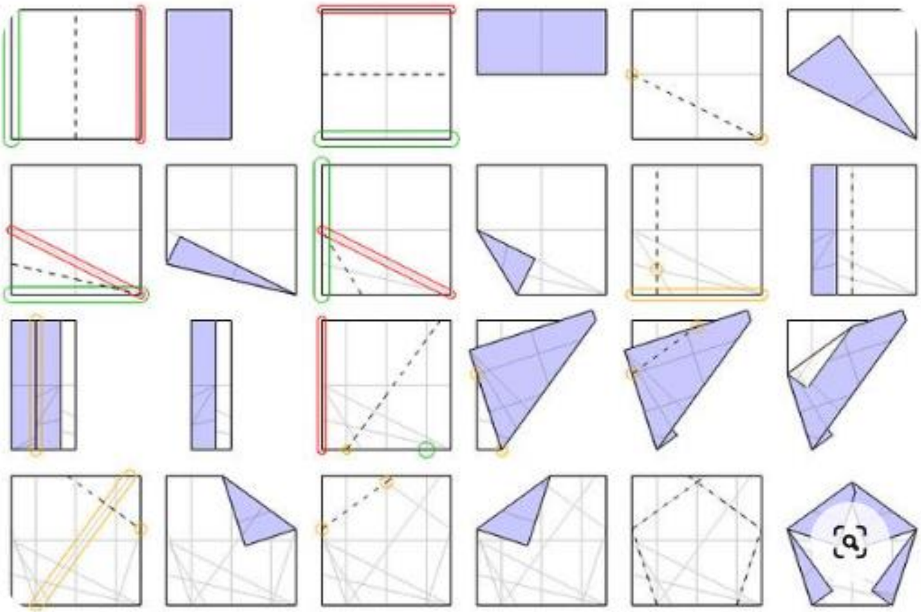
As same as classical construction, not all regular polygons could be folded by origami. To identify what regular n-sided polygon could be folded, mathematicians also formed a condition. A regular n-sided polygon can be constructed with origami if and only if n satisfy

$$n = 2^a 3^b \prod p_i \quad a, b = 0,1,2,3 \ldots$$

Where p_i is a Pierpont prime instead of a Fermat prime. A Pierpont prime is a prime satisfy the form

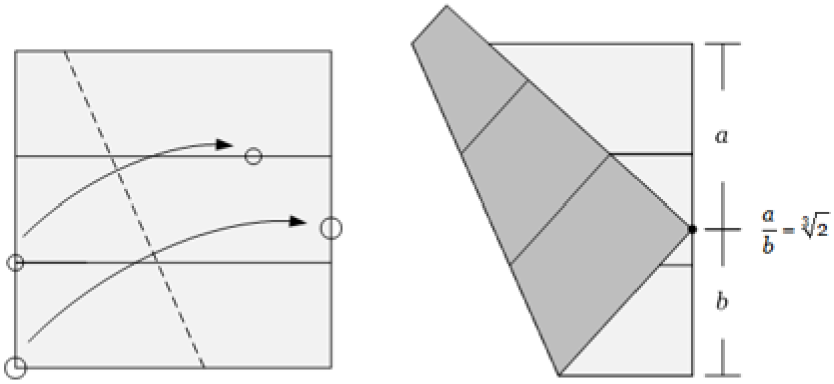
$$2^u 3^v + 1 \quad u, v = 0,1,2,3\ldots$$

Each origami regular polygon are complex to fold and really distinct, the following is an example of using origami to form a pentagon.

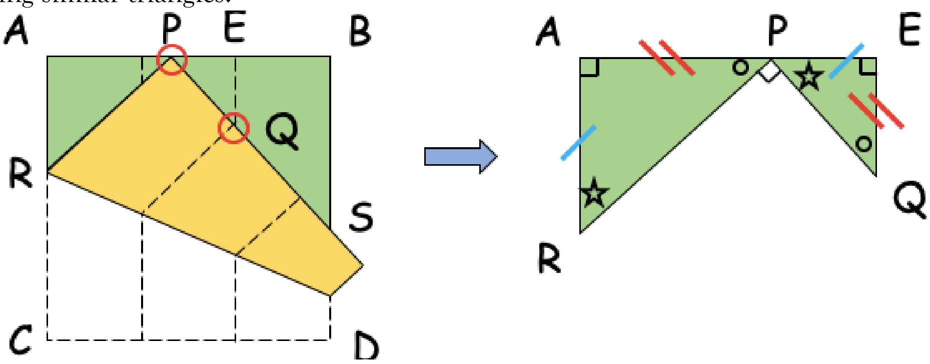


4.3 Solving the cube duplication problem

For solving the cube duplication problem, you cannot construct $\sqrt[3]{2}$ classically. Moreover, $\sqrt[3]{2}$ could be constructed through origami. In 1986, Peter Messer a way for origami cube-doubling.



Through dividing a square paper into three equal parts and folding the edges to the corresponding lengths as shown we can form $\sqrt[3]{2}$ easily. This technique can be proved through comparing similar triangles.



By setting APas unit length 1, BP as length x and AR as k.

$$PR = CR = 1 + x - k$$

In ΔAPR, by Pythagoras' theorem,

$$PR^2 = AR^2 + AP^2$$

$$(1 + x - k)^2 = k^2 + 1$$

$$k = \frac{x^2 + 2x}{2x + 2} \qquad PR = 1 + x - k = 1 + x - \frac{x^2 + 2x}{2x + 2}$$

And since $\Delta APR \sim \Delta EQP$ (SSS) and the square is trisected,

$$\frac{PR}{PQ} = \frac{AR}{EP}$$

$$\frac{1 + x - \frac{x^2 + 2x}{2x + 2}}{\frac{1 + x}{3}} = \frac{\frac{x^2 + 2x}{2x + 2}}{\frac{2(1 + x)}{3} - 1}$$

Through simplifying,

$$x^3 + 3x^2 + 2x = 2x^3 + 3x^2 + 2x - 2$$

$$x^3 = 2$$

$$x = \sqrt[2]{3}$$

We have proven that the origami cube duplication has formed $\sqrt[2]{3}$ and origami constructed something that not constructable in classical construction.

Conclusion

Greek problems are set of geometry problems asked by ancient Greek mathematicians centuries ago. They are hard to solve during the days. But the hard work and passion of mathematicians made them nonstop on solving these problems and after 2000 years later, these problems are finally proved to be insoluble or given constructable conditions. Interesting ways of solving them like origami are also done by modern mathematicians. The curiosity of ancient Greeks and fabulous work of mathematicians like Gauss and Galois had amazed me and I am really excited and proud of there work. As mathematicians, we shall never stop asking questions and never stop finding the solution to them.

Thank you for reading this essay and I hope you enjoyed it.

Thank you so much!

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