Fourier Transform: a mathematical poem or a recipe book ?

Virtually everything in the world can be described via a waveform - a function of time, space or some other variable. For instance, sound waves, electromagnetic fields, the elevation of a hill versus location, the price of your favorite stock versus time, etc. The Fourier Transform gives us a unique and powerful way of viewing these waveforms. Rightly said by Fourier, a very French guy, so his name is pronounced a little wonky: "4 E yay"). The central idea of it is that All waveforms, no matter what you scribble or observe in the universe, are just the sum of simple sinusoids of different frequencies. Although the whole story of its discovery is quite fascinating, and I will get back to that later. But before that let us try to understand the meaning of this ugly(?) looking mathematical poem or a recipe book.

- **What does the Fourier Transform do? You give it your favorite pizza and it gives back you the recipe.**
- **How?** Run the pizza through filters to extract each ingredient.
- **Why?** Recipes are easier to analyze, compare, and modify than the pizza itself.
- **Can we get the pizza back?** Yup!
- **How do we get the pizza back?** Blend the ingredients.

The Fourier Transform changes our perspective from consumer to producer, turning *What do I have?* How *was it made?*

In other words: given me pizza, i will tell you the recipe.

Why? Well, recipes are great descriptions of pizza. You wouldn't share a pice by piece analysis, you'd say "It had pepper, cheese and corn". A recipe is more easily categorized, compared, and modified than the object itself.

Well, imagine you had a few filters lying around:

- Pour through the " Olive Oil. " filter. 1 oz (about 29.57 ml)of olive oil
- Pour through the " Mozzarella Cheese - 50 gms. filter. 50 gm of Mozzarella Cheese
- Pour through the "tomato" filter. 3 of tomato
- Pour through the "capsicum" filter. 3 capsicums.

We can reverse-engineer the recipe by filtering each ingredient. The catch?

- **Filters must be independent**. The mozzarella cheese filter needs to capture mozzarella cheeses, and nothing else. Adding more tomato should never affect the mozzarella cheese reading.
- **Filters must be complete**. We won't get the real recipe if we leave out a filter ("There were tomato too!"). Our collection of filters must catch every possible ingredient.
- **Ingredients must be combine-able**. Pizza can be separated and re-combined without issue (A cookie? Not so much. Who wants crumbs?). The ingredients, when separated and combined in any order, must make the same result

Let us get back to story time, In Fourier's day, everyone thought they already knew what integrals, functions, and infinite series were, but in reality it was all rather vague – 'I know one when I see one.' So when Fourier submitted his epoch-making paper, there were good reasons for the academy officials to be wary. They refused to budge, so in 1822 Fourier got round their objections by publishing his work as a book, Théorie analytique de la chaleur ('Analytic Theory of Heat'). In 1824 he got himself appointed secretary of the academy,thumbed his nose at all the critics, and published his original 1811 memoir, unchanged, in the academy's prestigious journal.

The Fourier Transform takes a specific viewpoint: **What if any signal could be filtered into a bunch of circular paths?** Whoa. This concept is mind-blowing, and poor Joseph Fourier had his idea rejected at first.

So a simplified version of the FT is,

- The Fourier Transform takes a time-based pattern, measures every possible cycle, and returns the overall "cycle recipe" (the amplitude, offset, & rotation speed for every cycle that was found).

The Fourier Transform finds the recipe for a signal, like our pizza process:

- Start with a time-based signal
- Apply filters to measure each possible "circular ingredient"
- Collect the full recipe, listing the amount of each "circular ingredient"

Without further adieu, the Fourier Transform of a function g(t) is defined by:

$$\mathscr{F}\{g(t)\} = G(f) = \int_{-\infty}^{\infty} g(t) e^{-2\pi i f t}\, dt$$

The result is a function of *f*, or frequency. As a result, G(f) gives how much power g(t) contains at the frequency *f*. G(f) is often called the spectrum of g. In addition, g can be obtained from G via the *inverse Fourier Transform*:

$$\mathscr{F}^{-1}\{G(f)\} = \int_{-\infty}^{\infty} G(f) e^{2\pi i f t}\, df = g(t)$$

This equation states that we can obtain the original function g(t) from the function G(f) via the inverse Fourier transform. As a result, g(t) and G(f) form a Fourier Pair:

There are also many recent applications of wavelets to medical imaging. Hospitals now employ several different kinds of scanner, which assemble twodimensional cross-sections of the human body or important organs such as the brain. The techniques include CT (computerised tomography), PET (positron emission tomography), and MRI (magnetic resonance imaging). In tomography, the machine observes the total tissue density, or a similar quantity, in a single direction through the body, rather like what you would see from a fixed position if all the tissue were to become slightly transparent. A two-dimensional picture can be reconstructed by applying some clever mathematics to a whole series of such ‘projections’, taken at many different angles. In CT, each projection requires an X-ray exposure, so there are good reasons to limit the amount of data acquired. In all such scanning methods, less data takes less time to acquire, so more patients can use the same amount of equipment. On the other hand, good images need more data so that the reconstruction method can work more effectively. Wavelets provide a compromise, in which reducing the amount of data leads to equally acceptable images. By taking a wavelet transform, removing unwanted components, and 'detransforming' back to an image again, a poor image can be smoothed and cleaned up. Wavelets also improve the strategies by which the scanners acquire their data in the first place. In fact, wavelets are turning up almost everywhere. Researchers in areas as wide apart as geophysics and electrical engineering are taking them on board and putting them to work in their own fields. Ronald Coifman and Victor Wickerhauser have used them to remove unwanted noise from recordings: a recent triumph was a performance of Brahms playing one of his own Hungarian Dances. It was originally recorded on a wax cylinder in 1889, which partially melted; it was re-recorded on to a 78 rpm disc. Coifman started from a radio broadcast of the disc, by which time the music was virtually inaudible amid the surrounding noise. After wavelet cleansing, you could hear what Brahms was playing – not perfectly, but at least it was audible. It's an impressive track record for an idea that first arose in the physics of heat flow 200 years ago, and was rejected for publication