How to prepare the mathematically perfect vada? - Pratyush Manchandia

Growing up, one of my favourite foods were vadas. Whether they were store-bought, prepared in a restaurant or freshly fried at home I would eat them with gusto. However, as good as vadas prepared by others were, nothing could compare to the homemade vadas prepared by my mother. They were the closest thing to perfection, crispy on the outside and soft and warm on the inside. When I asked her how she made them so amazingly well, she replied, “They’re made with love-each and every one of them”. Now being a teenager, I simply couldn’t take that as an answer. So, I decided to embark on the journey to research what my mother did to make the most mouth-watering vadas.

For those who are unfamiliar with vadas, they are the South Indian version of a savoury donut. A lentil-based batter is deep fried to form a crispy outer crust whilst the inside remains soft and fluffy. It is usually served with a chutney and my mother would spread a thin layer of ghee (butter) on top of the vada to make the crunch ever more satisfying.


Figure 1

The vada can be modelled as a mathematical solid of revolution- the torus. For this essay, the torus will be parameterised. Of course, a vada won’t actually be a perfect circle but we’re making this assumption for ease of calculations.

Describing the torus:
The torus can be thought of as one core circle of radius R with a subsidiary circle of radius a revolved perpendicularly upon the core circle’s edge. This can be seen below in Figure 2.

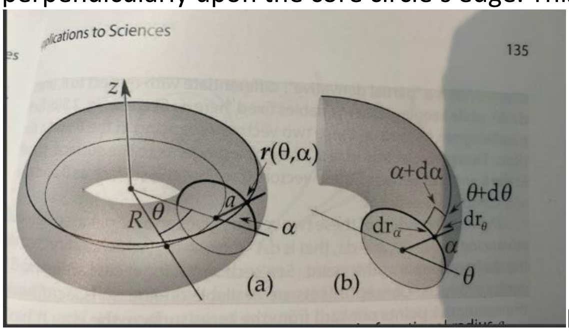

Figure 2

Moreover, two parameters are required to signify the two circles, namely θ and α. This can be seen in Figure 2(a) in the diagram above. As such any point on the surface of the torus can be described using the parameters listed above.

By positioning the torus as above, the centre at the origin and the core circle is aligned perpendicular to the z-axis. The core circle can be parametrically expressed in the form:
$(x(\theta), y(\theta), z(\theta))$ where $x(\theta)$, $y(\theta)$ and $z(\theta)$ are all functions of the parameter θ which is defined in the interval $(0, 2\pi)$.

By considering the core circle as simply a circle with radius R by definition:

$x(\theta) = R\cos(\theta)$
$y(\theta) = R\sin(\theta)$
$z(\theta)=0$

These initial set of equations represent a point on the circumference of the core circle. Since the centre of the subsidiary circle is placed on the circumference of the core circle, every point on the torus can be expressed as a displacement vector from the centre of the core circle.

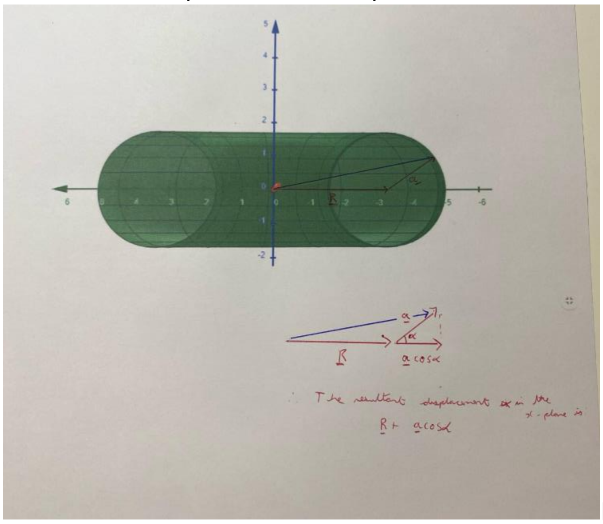
Figure 3

As shown in Figure 3, which represents a cross-section of the torus, the displacement vector to reach the surface of the torus is $R+a\cos(\alpha)$. Therefore, every point on the surface of the torus can be parametrically expressed in the form:
t = $(x(\theta,\alpha), y(\theta,\alpha), z(\theta,\alpha))$
As considered in figure 3, the displacement in the x-plane is $R + a\cos(\alpha)$. However, this displacement vector still lies within the core circle. Therefore, this displacement vector is within the parametric equation for the core circle. Combining these two compositely:

$(x(\theta,\alpha) = (R+a\cos(\alpha))\cos(\theta)$

Similarly, in the y-plane, the displacement vector will still be the same since the same vector is used to reach the surface of the torus. However, since the y-coordinate is required, the respective component of the parametric equation of the core circle is used. Therefore:

$y(\theta,\alpha)= (R+a\cos(\alpha))\sin(\theta)$

Finally, in the z-plane, the displacement is independent of the core circle since it is positioned perpendicular to the z-axis. As a result, the only thing affecting the z-component is the vertical component of the subsidiary circle's radius vector. Therefore:

$z(\theta,\alpha))= a\sin(\alpha)$

Putting this all together, the parametric equation to describe a point on the surface of the torus is:
r = $((R+a\cos(\alpha))\cos(\theta), (R+a\cos(\alpha))\sin(\theta), a\sin(\alpha))$

Now transferring back to the context of the vada, we have found the equation for any point on its surface. This immediately won't provide us with much use on how to prepare the vada. A natural place to start with is finding out how much batter is required to prepare a vada. We will need to find the volume of the torus!

Volume of a torus:
There are many methods to find the volume of the torus, including the more mathematical calculus approach. However, instead I find that by considering the problem carefully, an easier, more intuitive solution can be found.

Consider a thin slice of a subsidiary circle of thickness dC. Then, the volume of the thin slice will simply be the area of the circle times the thickness. As seen on Figure 4, this is simply:
$\pi a^2 \, dC$.

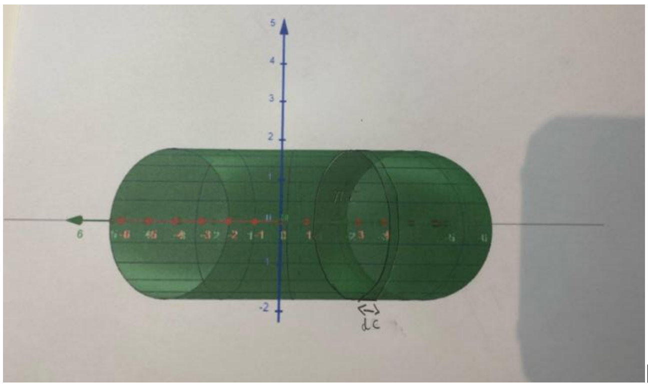

Figure 4

Thinking carefully about the total length of the thickness, we get that the volume should simply be $C(\pi a^2)$ where C is the circumference of the core circle. Substituting this in, we get the volume of the torus to be $2\pi^2 a^2 R$.

This can be confirmed using calculus. Since the volume of one thin slice was found out to be $\pi a^2 \, dC$, if we sum all of these little slices, we should obtain a value for the volume of the torus.
Imagine unfolding the torus so that it forms a cylindrical shape; the length of the torus would be the circumference of the core circle. Therefore, this can be represented as:

$\int \pi a^2 \, dC$
$\Leftrightarrow \pi a^2 C + c$

Which yields the same result as before. (note the +c term is ignored since the integral would usually be definite)

Therefore, the total volume of batter required to make a vada is $2\pi^2 a^2 R$. By inputting some typical values for a vada:

$a \approx 0.015\text{m}$

$R \approx 0.04\text{m}$

Therefore, the volume of batter required would be:

$2\pi^2(0.015)^2(0.04) \approx 1.8\times10^{-4}\ \text{m}^3$

≈ 0.18 litres (This seems like a reasonable estimate)

Now a vada wouldn't be complete without the soft butter on top which provides a contrasting texture. For this, the surface area of the torus would have to be worked out. Again, there are many methods to find the surface area. The simplest being simply differentiating the volume. However it isn't immediately obvious what we're differentiating the volume with respect to; there are two variables, namely a and R.

This can also be conducted using calculus to find the surface area. This method is rather tedious but nevertheless provides the right answer which could be useful for the confusion above:

First, we define little increments on the surface of the torus as $d\mathbf{r}_\theta$ and $d\mathbf{r}_\alpha$ where:

$$d\mathbf{r}_\theta = \frac{\partial \mathbf{r}}{\partial \theta} d\theta$$

$$d\mathbf{r}_\alpha = \frac{\partial \mathbf{r}}{\partial \alpha} d\alpha$$

By partially differentiating each component of the parametric equation, with respect to θ and then with respect to α, the following equations can be derived:

$$d\mathbf{r}_\theta = (x(\theta,\alpha), y(\theta,\alpha), z(\theta,\alpha))$$

where:

$$(x(\theta,\alpha) = \frac{\partial}{\partial \theta}\big((\mathrm{R} + \mathrm{a}\cos(\alpha))\cos(\theta)\big)d\theta$$

$$= -(R+a\cos(\alpha))\sin(\theta)\, d\theta$$

$$(y(\theta,\alpha) = \frac{\partial}{\partial \theta}\big((\mathrm{R} + \mathrm{a}\cos(\alpha))\sin(\theta)\big)d\theta$$

$$= (R+a\cos(\alpha))\cos(\theta)\, d\theta$$

$$(z(\theta,\alpha) = \frac{\partial}{\partial \theta}(a\sin(\alpha)d\theta$$

$$= 0$$

Therefore, $d\mathbf{r}_\theta = (-(R+a\cos(\alpha))\sin(\theta)\, d\theta, (R+a\cos(\alpha))\cos(\theta)\, d\theta, 0))$

Similarly, $d\mathbf{r}_\alpha = (x(\theta,\alpha), y(\theta,\alpha), z(\theta,\alpha))$

where:

$(x(\theta,\alpha) = \frac{\partial}{\partial\alpha}\big((\mathrm{R} + \mathrm{a}\cos(\alpha))\cos(\theta)\big)d\alpha$

$= -\sin(\alpha)\cos(\theta)\ d\alpha$

$(y(\theta,\alpha) = \frac{\partial}{\partial\alpha}\big((\mathrm{R} + \mathrm{a}\cos(\alpha))\sin(\theta)\big)d\alpha$

$= -\sin(\alpha)\sin(\theta)\ d\alpha$

$(z(\theta,\alpha) = \frac{\partial}{\partial\alpha}(\mathrm{a}\sin(\alpha)d\alpha$

$= a\cos(\alpha)\ d\alpha$

Therefore, $d\mathbf{r}_\alpha = (-\sin(\alpha)\cos(\theta)\ d\alpha, -\sin(\alpha)\sin(\theta)\ d\alpha, a\cos(\alpha)\ d\alpha))$

The formula for the SA is:
$\iint_D |d\mathbf{r}_\theta \times d\mathbf{r}_\alpha|\ d\theta\ d\alpha$

Therefore finding the vector product of the two tangential vectors will result in the area vector. Through some rigorous algebra the magnitude of the cross product can be found to be:

$a(R+a\cos(\alpha))\ d\theta\ d\alpha$

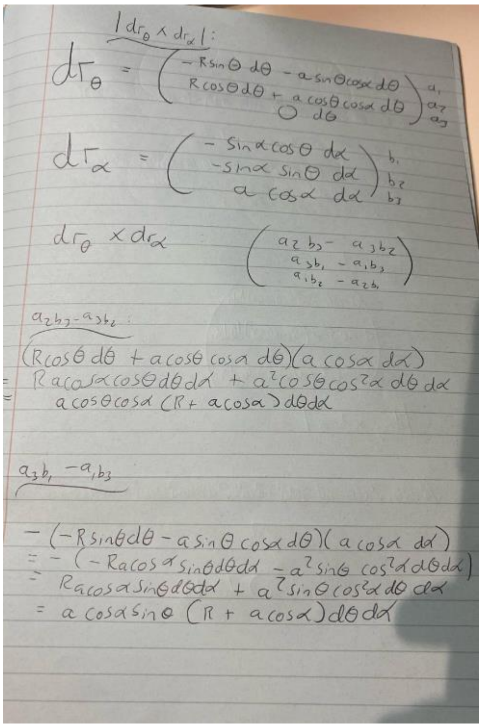

Figure 5(a)

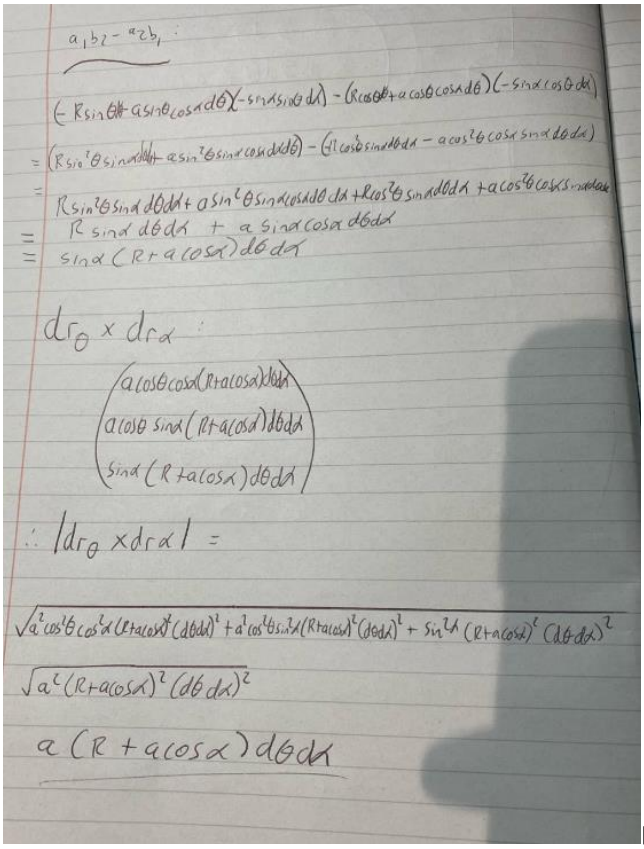

Figure 5(b)

This can then be integrated. The bounds on the integrals are the domain of the parameters- 0, 2π.

$\iint_D a(R+a\cos(\alpha))\, d\theta\, d\alpha$
⇔$\boldsymbol{a} \int_{\mathbf{0}}^{2\pi} d\theta \int_{\mathbf{0}}^{2\pi} R + a\cos(\alpha))\, d\alpha$
⇔a (2π) ($2\pi R$))
⇔4$\pi^2 aR$

Therefore, going back to the confusion before, we are differentiating with respect to a. Thinking about this it seems to make sense; when increasing the surface area, you increase the radius a as this has the same effect as adding on a small layer on top and increasing the surface area.

With this result, the volume of ghee(butter) can be estimated:
For ease, we assume that the butter is distributed uniformly on the surface. We let the thickness of the butter be denoted as dB. Then, the volume of this layer can be found out:

$V = 2\pi^2(a+dB)^2R - 2\pi^2a^2R$
By noting the similarities to first's principle, we deduce:
$f(a) = 2\pi^2a^2R$
therefore, $V = f'(a)\, dB$
$V = 4\pi^2 aR\, dB$

This result suggests simply that the volume of the thin layer is simply the surface area multiplied by the thickness. This is the Pappus's centroid theorem, and it is an interesting application of it.

Now for the juiciest part; the crispiness formula. The crispiness can be thought off as a layer on the subsidiary circle.

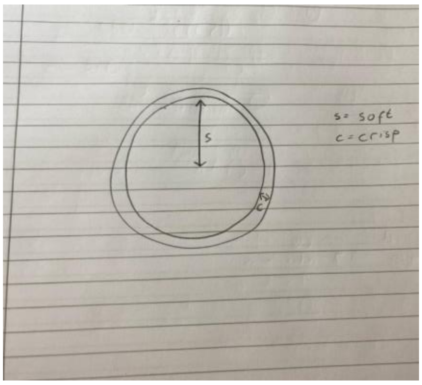

Figure6

Therefore, a = s + c

The volumes of the respective parts are:

Soft:

$2\pi^2R(s)^2$

Crisp:

$2\pi^2R(s+c)^2 - 2\pi^2R(s)^2$

$\Leftrightarrow 2\pi^2R(s^2 + c^2 + 2cs – s^2)$

$\Leftrightarrow 2\pi^2R(c^2 + 2cs)$

The soft: crisp ratio is therefore:

$\frac{s^2}{c(c+2s)}$

We call this ratio y. We also know that s + c = a

Therefore,

$y = \frac{s^2}{(a-s)(a+s)}$

$\Leftrightarrow y = \frac{s^2}{a^2-s^2}$

$\Leftrightarrow \frac{dy}{ds} = \frac{(a^2-s^2)(2s)-(s^2)(-2s)}{(a^2-s^2)^2}$

$\Leftrightarrow \frac{dy}{ds} = \frac{2sa^2}{(a^2-s^2)^2}$

Therefore, there is a stationary point at s=0. Finding the second derivative to work out the nature of the stationary point:

$\Leftrightarrow \frac{d^2y}{ds^2}= \frac{((a^2-s^2)^2)(2a^2)-(2sa^2)(4s^3-4a^2s)}{((a^2-s^2)^4)}$

At s=0, $\frac{d^2y}{ds^2} = 2a^{-2}$

Since a represents the radius and a > 0 $\forall a$, $\frac{d^2y}{ds^2} > 0$*. Therefore, the point s=0 is a minimum. At s=0 (a=c), y=0 therefore, this donut represents maximum crisp.*

However, there seems to be no maxima. There is, however, a vertical asymptote at the point where $a^2 - s^2 = 0$*. This is where a=s. Therefore, as s→a, y→∞. Therefore, there is minimum crisp at the point as s→a.*

The crispiness of the vada is independent of the radius of the core circle and instead relies on only the subsidiary circle.

Therefore, by substituting some measured values for a, s and c, we can find out the common ratio for vada.

a = 0.015m

s= 0.012m

c= 0.003m

Therefore, the corresponding y value is, 1.8.

In conclusion, it seems that there is a lot of maths behind crafting the perfect vada that my mother cooks. It seems highly unlikely that she calculated all this before she fried them; it seems to come from intuition. The key thing to appreciate is the beauty of maths; how it pops up everywhere and to ENJOY VADAS.

Appendix:

Figure 1: Homemade vada with dimensions:

R = 0.04m

a= 0.015m

s= 0.012m

c= 0.003m

Figure 2: Short extract from Maths for Sciences book.

Figure 3 and 4: 3D model on GeoGebra

Thank you to my mother and Isaac Physics for inspiring this essay.