Group Theory and it's applications in Rubik's cubes

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1 Introduction

What is a group? This field is all about symmetry where performing some action on the object preserves the group structure. Consider the symmetries of a cube, there are 24 rotations on a cube that leave it looking the same, notice that we drop some information of the group structure such as the colour on each side as otherwise we wouldn't call the cube symmetric. Another example of dropping of information of the cube is considering the faces of the cube to be able to move independently, we could think of this as the the number of symmetries of the centre pieces of the Rubik's cube because they never move. This group preserves much less structure so the the number of symmetries is much higher.

In this essay we formally develop this idea of dropping information to more accurately describe the symmetries preserved in the Rubik's cube, and show how understanding this group structure is useful by constructing algorithms to solve the cube.

2 Group fundamentals

2.1 what are groups really?

A group is a more abstract concept as an algebraic structure satisfying some specific axioms:

Definition 1 (Group Axioms)**.** A *group* is a set G binary operation * satisfying the following axioms:

1. There is some e $\in G$ such that for all a, we have

$$a * e = a = e * a \qquad \text{(identity)}$$

2. For all $a \in G$, there is some $a^{-1} \in G$ such that

$$a * a^{-1} = e = a^{-1} * a \qquad \text{(inverse)}$$

3. For all $a, b, c \in G$, we have

$$(a * b) * c = a * (b * c) \qquad \text{(associativity)}$$

This looks rather intimidating so lets see if we can relate this to symmetries of objects which we are for familiar with. However lets first remind ourselves with what a symmetry is.

Definition 2. A *symmetry* is something we do to an object which leaves the object intact. (symmetry)

Example 1. consider the rotational symmetries of an equilateral triangle, rotations by 0°,120° and 240°. What is important is that we don't require that

the symmetry leaves *everything* intact, otherwise we wouldn't be able to do anything, what we do care about is how the resulting object looks, but we don't care about where the individual vertices went.

We can now see how our intuitive ideas of symmetry relate to the group axioms.

On our equilateral triangle a rotational symmetry, is 0° which does nothing to our triangle not even our vertices are changed! This is our corresponding identity symmetry. We can also easily find our inverse symmetry, consider doing a 120° rotation followed by a 240° rotation. In total, the rotation of the triangle is 360°, which is equivalent to 0°, hence 240° and 120° are inverse elements of each other.

Motivation for associativity is a little trickier but we can consider the rotations on our equilateral triangle equivalent to function composition, which by definition is associative.

As the rotational symmetries follow the *group axioms*, they evidently form a **group of symmetries**, where the rotations 0° the identity element ,120° 240° correspond to the *group elements*.

So we can now more formally define symmetries as groups which have the following properties:

1. Composition of symmetries is a symmetry.
2. Composition of symmetries is associative.
3. Not doing anything is a symmetry.
4. We can undo symmetries with inverse elements.

2.2 Rubik's Cube Rules

pieces of the Rubik's Cube

In this section we will define how we think of the individual pieces of the Rubik's cube and how we notate moves we perform on the Rubik's cube.

The 3x3 Rubik Cube is a cube composed of 3 smaller cubes in each direction, so we would expect it to contain 27 individual cubes, however when taking the cube apart we can see that the centre cube does not exist, so the Rubik cube is only composed of 26 smaller cubes.

We call the individual cubes of the Rubik's Cube *Cubies* and they are split into 3 categories: corner, edge and centre cubies. *from now on we will refer to the Rubik's Cube as cube, as well as Rubik's Cube.*

1. Corner cubies have 3 visible faces, there are 8 of them on the cube.
2. Edge cubies have 2 visible faces, there are 12 of them on the cube.
3. Centre cubies have 1 visible face, there are 6 of them on the cube.

Each cubie lives in a specific place in the solved state of the cube, which we call the *cubicle*, and for the cubie to be in the cubicle the cubicle needs to have the correct *position* and *orientation*.

We shall note: that the Rubik's Cube always keeps the centre cubies in their cubicles, as the centre cubies **do not move**.

And that: each piece on the Rubik's cube are **distinct**, meaning an edge cubie cannot become a corner cubie vice versa.

Moves on the Rubik's Cube

We shall now note the notation used to denote the basic moves on the Rubik's Cube, These refer to specific group elements of what we will construct to be the Rubik's Cube Group.

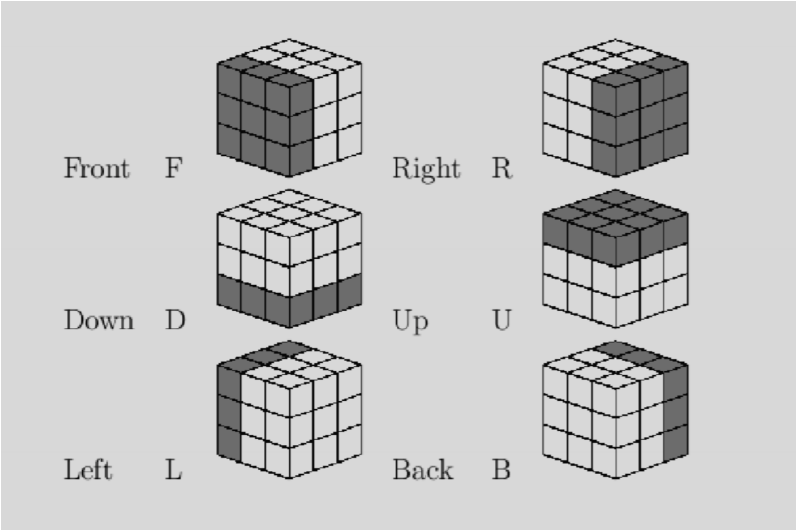

Figure 1: Basic moves on a Rubik's Cube, shows the corresponding the name, symbol and image of each basic move on the Rubik's Cube.

Each basic move corresponds to the face of the Rubik's cube being rotated 90 ° clockwise.

2.3 Why is the Rubik's Cube a group?

In this section we show that the Rubik's Cube is a group, and the associated elements and group actions of the group.

In order for something to be a group it must satisfy our group axioms **Definition 1**, however we can use our Symmetry axioms to verify the Rubik's Cube is a group more intuitively.

1. Composition of symmetries is a symmetry.

This is intuitively true any general moves M_1, M_2 where M_1 is performed first then M_2, $M_1 * M_2$ is obviously also a move for any general move. 2.Composition of symmetries is associative.

This needs proving.

3.Not doing anything is a symmetry.

Let the "do nothing move" represent the group element e, for all general moves M_1, "doing nothing" then M_1 is the same as doing M_1 then "doing nothing" and the same as just doing M_1.

4. We can undo symmetries with inverse elements.

Suppose we have the solved cube and are just rotating the front face, using F repeatedly if we rotate the front face 4 times *denoted* F^4 then undo our last clockwise rotation, we get F^3 however rotating the front face 4 times leaves our cube in the solved state, our initial position, so rotating front face 4 times is the same as the "do nothing move", hence F^4=e (1), and undoing our last clockwise rotation, we can think of as F^-1 applying this to our equation (1) gives us $F^3 = F^-1$, as each basic moves is the same as rotating the corresponding face clockwise, we have an associated inverse element for each basic move and as we have symmetry composition by 1, composition of each basic move represents and general move, each general move M_1 has a corresponding inverse M_1^{-1} hence we can undo symmetries with inverse elements.

Hence the set of moves of our Rubik's Cube is a group which we shall denote $(\mathfrak{R}, *)$

2.4 Properties of the Rubik's Cube Group

First we shall define some group properties

Definition 3 (Order of a group)**.** The order of the group G, denoted by $|G|$, is the number of elements in G.

The order of the *Rubik's Cube Group* is $|\mathfrak{R}|$=43,252,003,274,489,856,000, this is why we only work with the 6 basic moves as studying a group this size is unrealistic.

To help deal with the size of the group we shall split the group in smaller groups and study those groups individually. We call these smaller groups subgroups.

Definition 4 (Subgroup)**.** A non-empty subset H of a group $(G, *)$ is called a subgroup of G if $(H, *)$ is also a group.

Example 2. The group of rotational symmetries of a square is a subgroup of $\mathfrak{R}$, why? well we can consider it to be the same as the rotations *generated* by just applying F on the cube repeatedly. *We call the sameness of two groups an isomorphism between the groups.*

In-fact this subgroup is a special kind of subgroup called a cyclic subgroup.

Definition 5 (Cyclic groups)**.** A group G is called cyclic if and only if for all $g \in G$ every element in G is some power of g.

We call the specific element g with this property the generator of G. Where the set $\langle g \rangle = \{e, g, , g^2, \dots g^{n-1}\} = G, \quad n = |G|$

we can also have generators which are subsets of the corresponding group, Let H be a group S be a subset of H, S generates H if $H = \langle S \rangle$

We shall now use this result to state the following theorem.

Theorem 1. Let H be a finite group and S be a subset of G. Then, $G = \langle S \rangle$ if and only if every element of G can be written as a finite product of elements of S.

Proof. needs proving. □

Example 3. Every element of $\mathfrak{R}$ can be written as a sequence of elements of the Rubik's Cube's basic moves, so the set of the basic moves generates the Rubik's cube Group! $< F, D, L, R, U, D >= \mathfrak{R}$.

So now we have a more convenient way of studying the group, by studying the basic moves.

We also have the property that the Rubik's cube is non-abelian meaning that the order of our moves matter $RF \neq FR$, and is also what contributes to the Rubik Cube Group being so large. Often we will study abelian properties of our subgroups to make understanding the Cube more manageable.

3 The symmetric Group

We defined symmetries as operations which leave the object intact, in this section the symmetric group is when we don't care about the group structure at all. We will use this to see how the Rubik's Cube Group acts on specific Cubies.

3.1 Homomorphisms

first we will look at Homomorphisms, these are functions between groups that allow some elements of a group to be preserved and everything else to be lost.

Definition 6 (Homomorphisms)**.** Let $(G, *)$ and $(H, \times)$ be groups. A function $f : G \to H$ is a group homomorphism if and only if

$$\text{for all } g_1, g_2 \in G \quad f(g_1 * g_2) = f(g_1) \times f(g_2)$$

Note: we don't have any requirement that the function f needs to uniquely map every element of G onto a different element of H nor that every element in H needs to have a corresponding element in G. If this was the case then f would describe an isomorphism and G, H would have the same binary operator.

This means our co-domain H can be smaller than G which is the property that will allow us to lose group structure under the function f.

We can't just take a piece of our information of domain and use that to collectively study our group, you could imagine that the missing piece contradicts our chosen piece, so it's important to have holistic view of the group which only allows us to discard information in a systematic way.

Example 4. Lets consider the rotational symmetries of our equilateral triangle again, but now we will label the vertices 1, 2, 3, if we send the vertices $1 \to 2$ and $2 \to 1$ is this a rotational symmetry? No, but it is still a type of symmetry, specifically a flip in the line through vertex 3.

We call this type of symmetry a permutation acting on the vertices.

Definition 7 (Permutation)**.** A permutation of the set X is a bijection from a set X to X itself. The set of all permutations on X is Sym X.

If the set X is finite and has n elements $|X| = n$ then we denote Sym X as S_n We can think S_n as the collection all the ways of arranging the set X where order matters so $|SymX| = n!$. *we have a bijection between two sets so we can think of the elements list placement encoding which element it maps to.*

3.2 Rubik's Cube cubies as symmetric groups

We shall recall that the Rubik's Cube is composed of 26 cubies, of which 6 centre cubies do not move, So under any move the centre piece is always sent to there cubicles so we can ignore any permutations on the centre cubies. We shall label these cubies $\{1, 2, 3, \ldots, 20\}$ in some way. After some general move M we get the corresponding permutation P_{cube}. By defining a homomorphism ϕ_{cube} to the symmetric group S_{20} by

$$\phi_{cube} : \mathfrak{R} \to S_{20}, \quad \phi_{cube}(M) = P_{cube}$$

Similarly we can define a homomorphism to S_8 for the corner cubies $\phi_{corner} : \mathfrak{R} \to S_8$ and the homomorphism to S_{12} for the edge cubies $\phi_{edge} : \mathfrak{R} \to S_{12}$.

As these are both homomorphisms we need it to satisfy Definition 6, so as the Symmetric group has binary operator of function composition, where the inside function is evaluated before the outer function we have for ϕ_{cube}.

$$\phi_{cube}(M_1 * M_2) = \phi_{cube}(M_2) \circ \phi_{cube}(M_1)$$

3.3 Cycle notation

Cycle notation gives us a way to write out elements of the symmetric group, with each element telling how the cubies permute.

Remember that we denote cubies as the numbers from $1, \ldots, n$ We have two notations for cycle notation:

Notation 1. we write $1, \ldots, n$ on the top line and the images below, consider a general element in S_3, σ if $\sigma : X \to X$, we write:

$$\begin{pmatrix} 1 & 2 & 3 \\ \sigma(1) & \sigma(2) & \sigma(3) \end{pmatrix}$$

$$S_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right. \\ \left. \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \right\}$$

This notation is rather cumbersome, it also is a bit difficult to tell that some elements of the group cycle, the cubies.

consider the group element, $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ which we will denote τ.

$\tau : X \to X$ is a map that sends $1 \mapsto 3,\ 2 \mapsto 1,\ 3 \mapsto 2$.

To avoid writing cubies multiple times we could denote this as $1 \mapsto 3 \mapsto 2 \mapsto 1$ where we have a more obvious cycle that repeats, this is our motivation for our second notation.

Notation 2. For our element τ we write this in cycle notation by showing where each cubie maps to for the first sequence of the cycle $(1 \mapsto 3 \mapsto 2)$.
It is common to omit the '$\mapsto$' for simplicity: $\tau = (1\ 3\ 2)$.

Consider the group element $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$ as $2 \mapsto 2$ we leave it out and thus write (1 3) for the cycle notation of the group element where we denote the smallest numbered cubie first for convention.

Example 5. We can now denote S_3 more conveniently in cycle notation as

$$S_3 = \begin{Bmatrix} e & (123) & (132) \\ (12) & (13) & (23) \end{Bmatrix}$$

3.4 Properties of Cycles and Cube Moves

In this section we will state properties of Cycles.

Definition 8 (k-Cycles and transpositions)**.** We call $(a_1\ a_2\ a_3 \dots a_k)$ a k-cycle, where k is the length of cycle, the number of cubies in the cycle.

A transposition is a 2-cycle, specifically.

Example 6. Every basic corner move has a 4-cycle, consider the permutation $\phi_{corner}(F) = (2376)$, this 4-cycle is due to each basic move being isomorphic to the rotational symmetries of a square.

But we aren't limited to 4-cycles only on the cube, consider the Move RF, we want to know where the corners go when we apply R first then F:

$$\begin{aligned} &\phi_{corner}(RF) = \phi_{corner}(F) \circ \phi_{corner}(R), \\ &\text{and,} \quad \phi_{corner}(R) = (1265), \quad \phi_{corner}(F) = (2376) \\ &\text{so,} \quad \phi_{corner}(RF) = (2376) \circ (1265) \end{aligned}$$

Cycle composition multiplication

Recall the composition goes from right to left, where we start with the smallest cubie, so $1 \mapsto 3$ because in (1 2 6 5) $1 \mapsto 2$, and (2 3 7 6) further maps it to 3 as $(2 \mapsto 3)$. Then we pick the next smallest cubie and repeat. So we should get in total $(1 \mapsto 3), (3 \mapsto 7), (6 \mapsto 5)$ and $(5 \mapsto 1)$ This mapping is cyclic so we can express it as a 5-cycle.

Hence $\phi_{corner}(RF) = (13765)$ a 5-cycle, this gives us the intuitive idea that our cycles become more complex when we compose more basic moves.

Proof of non-commutativity of Rubik cubes

Lets also consider $\phi_{corner}(FR) = (1265) \circ (2376)$ notice how we've swapped the order of the cycles this becomes $\phi_{corner}(FR) = (12375)$ Notice that $(13765) \neq (12375)$ this shows that $RF \neq FR$, Our proposition in Example 3. However this is not true in general as some specific moves are commutative.

Consider Moves RL and LR where $\phi_{corner}(L) = (3487)$
$\phi_{corner}(RL) = (3487) \circ (1265) = (1265) \circ (3487) = \phi_{corner}(LR)$

why?, well first thing we notice is that both R and F cycles have no cubies in common, we call two cycles with this property disjoint.

Properties of the support and disjoint cycles

We can more formally define what disjoint cycles are if we we first define the support of a cycle.

Definition 9 (Support)**.** We call the support of move the set of numbers which are cycled by that move σ , denoted $\text{supp}(\sigma)$

The support of (3 4 8 7) is $\{1, 3, 4, 7\}$

We can now define the disjoint of two cycles similar to this disjoint of two sets.

Definition 10 (Disjoint)**.** Two cycles σ, τ are disjoint if they don't have any cubies in common $\text{supp}(\sigma) \cap \text{supp}(\tau) = \emptyset$

Now we can formally prove that disjoint cycles commute:

Lemma 1. Disjoint cycles commute.

Proof. Our method is that we want to show that any general element i is preserved or a function i is preserved for both $\sigma \circ \tau$ and $\tau \circ \sigma$ meaning that for all i, $\sigma \circ \tau = \tau \circ \sigma$

As cycles σ, τ are disjoint we have two cases for the inclusion of general element i:

1. 1 $i \notin \text{supp}(\sigma)$ and $i \notin \text{supp}(\tau)$, in this case $\sigma(i) = i$ and $\tau(i) = i$ as σ, τ have on effect on i so we have:

$$(\sigma \circ \tau)(i) = \sigma(\tau(i)) = \sigma(i) = i = \tau(i) = \tau(\sigma(i)) = (\tau \circ \sigma)(i)$$

2. 2 Otherwise i is in the support of just σ or τ. without loss of generality we will consider $i \notin \text{supp}(\sigma)$ and $i \in \text{supp}(\tau)$. In this case $\tau(i) = i$, so $(\sigma \circ \tau)(i) = \sigma(\tau(i)) = \sigma(i)$, now as σ, τ are disjoint $\sigma(i) \notin \text{supp}(\tau)$ so $(\tau \circ \sigma)(i) = \tau(\sigma(i)) = \sigma(i)$ this implies $(\sigma \circ \tau)(i) = (\tau \circ \sigma)(i)$

As both of these cases hold for general i they hold for all $i \in \{1, \ldots n\}$ □

Lemma 2. we can write every permutation as a composition of disjoint cycles.

Proof. Here's an intuitive way we can think about this if we consider a single cycle to be disjoint, so we have 2 or more disjoint compositions by this lemma. As we have the symmetric group we have a bijection between $\{1, \ldots, n\}$ and $\{1, \ldots, n\}$ so every element must be mapped onto so the largest possible case of compositions is all 2-cycles where they are all disjoint, every other case is a composition of k-cycles, where $k \geq 2$. □

Each permutation has a unique disjoint cycle decomposition up to re-ordering, we call this the permutations cycle type.

Definition 11. (Cycle Type) If $\sigma \in S_n$ is the product of disjoint cycles of lengths $n_1, \ldots, n_r$ (including 1-cycles), then the integers $n_1, \ldots, n_r$ are called the cycle type of σ.

Properties of transpositions

Lemma 3. Every permutation is a composition of transpositions.

Proof. recall from lemma 2 that every permutation can be written as a composition of disjoint cycles, so it suffices to show that each cycle can be written as a composition of transpositions.

This fact arrives intuitively from that The symmetric group that we have a bijection between two sets so each cubie has a 1-1 mapping, between two cubies, a transposition. □

Thus a k-cycle $(a_1\ a_2\ a_3 \ldots a_k)$ can be written as a composition of k-1 transpositions $(a_1 a_2)(a_2\ a_3) \ldots (a_{k-1} a_k)$.

Note that the number composition of transpositions is not unique
$(12345) = (12)(23)(34)(45) = (12)(23)(12)(34)(12)(45)$.

in-fact any permutation could have any multiple of 2 transposition compositions added onto the permutation where its permutation is not effected.

Definition 12 (Even and Odd permutations)**.** A permutation σ is even if it is has a 2n cycle decomposition of transpositions and odd if it has 2k+1 transposition cycle decomposition, $k, n \in \mathbb{N}$.

We now want to show how we can preserve the structure of even and odd cycles under cycle composition.

as a single 2 cycle is an odd permutation, any even number of odd permutations makes an even permutation, and any odd number of odd permutations makes an odd permutation.

So it seems like regardless of if we have an even or order cycle we can always make an even permutation.

4 Alternating group

In this section we show how the the Rubik's cube in the alternating group

Definition 13. Alternating group A_n is all the even permutations in S_n.

Lemma 4. If $\sigma \in S_n$, then if and only if there are an even number of cycles of even length in its decomposition into a product of disjoint cycles.

Proof. We recall that an even length cycle 2k has 2k-1 compositions of transpositions in its cycle decomposition of transpositions which is odd, so we must have an even number of cycles so that the overall permutation is even. □

We may now prove that some cubie arrangements are not possible in S_n

Theorem 2 (If $M \in \mathfrak{R}$ then $\phi_{cube}(M) \in A_{20}$)**.**

Proof. It is sufficient to check this for M = U, D, L, R, F, B since any M may be written as a product of these. If M is one of these, then $\phi_{cube}(M)$ is a product of two four-cycles, namely an edge 4-cycle and a corner 4-cycle. Hence it is an even permutation, by the previous theorem. □

As single 2-cycles aren't in A_{20} that tells we cannot move only two cubies around the Rubik's Cube in a single move. This tells us our simplest move we can use is a 3-cycle as 2-cycles are impossible.

How do we solve the cube?

The Rubik's Cube Group is a very large group so we want to have moves that move the smallest number of cubies around to be able to tell what is going on when trying to solve the cub. So want some how to be able to measure how many pieces are exchanged by looking at a move, and which cubies are involved in that specific move.

5 The Commutator

5.1 what is the Commutator

In this section we will describe what a commutator is and how we will use it to find 3-cycles to help solve the cube.

the commutator combined with the support allows us to measure how commutable a move is.

Definition 14 (Commutator)**.** consider moves M_1, M_2 we call the move $M_1M_2M_1^{-1}M_2^{-1}$ the commutator of M_1 and M_2 denoted $[M_1, M_2]$.

We notice that the term commutator is very close to the term commutable, this is because the commutator has the commutable property:

$$[M_1, M_2] = e \quad \text{if and only if} \quad M_1M_2 = M_2M_1$$

we can see this by inserting $M_1M_2 = M_2M_1$ into the definition for commutator.

We recall that disjoint cycles commute, and cycles are disjoint when the support of M_1 and M_2 is $\emptyset$

so we have the relation:

$$\text{supp}(M_1) \cap \text{supp}(M_2) = \emptyset \iff M_1M_2 = M_2M_1 \iff [M_1, M_2] = e$$

if we take the contrapositive we have:

$$[M_1, M_2] \neq e \iff M_1M_2 \neq M_2M_1 \iff \text{supp}(M_1) \cap \text{supp}(M_2) \neq \emptyset$$

So we don't want our support of M_1 and M_2 to be empty but we do want it to remain as commutative as possible to have nice properties (why?).

5.2 Commutator Properties

We want to be able to have the result that when there is 1 cubie in the support of M_1, M_2 then $[M_1, M_2]$ is a 3 cycle.

Theorem 3 ($[\sigma, \tau] \in A_n$)**.** Let $\sigma, \tau \in S_n$ Then $[\sigma, \tau] \in A_n$

Proof. We shall not proof this result here as the result is rather technical, but we will assume it to be true for the next theorem. □

6 Conjugacy Classes

In this section we think of conjugation as a change of coordinates of the cubies.

6.1 What is conjugation?

Definition 15 (Conjugation of a move)**.** Conjugation of a move M_1 by M_2 is when apply M_2 to the left side and M_2^{-1} to right side of M_2 so the conjugation of $M_1 = M_2M_1M_2^{-1}$

Under the operation of conjugation we say that M_1 and $M_2M_1M_2^{-1}$ are similar to each other but not necessarily equal.

If you have studied linear algebra this formula bab^{-1} is used for the change-of-basis formula, where a is a transformation in our basis and b is the matrix which transforms our matrix to a different basis.

This idea is isomorphic to our conjugation of a Rubik's Cube move.

6.2 Equivalence classes

We said M_1 and $M_2M_1M_2^{-1}$ are similar but we want to measure how similar?

The way we can measure how similar is by using a relation.

Definition 16 (Relation)**.** Let $\sim$ be a relation on X, that is for every element $x, y \in X$, $x \sim y$ is either true or false.

Example 7. For conjugation we have the relation for $x, y \in S_n$ that $x \sim yxy^{-1}$

equivalence relation

Theorem 4. Two elements of s_n are only conjugate if and only if they have the same cycle structure.

Where the cycle structure is preserved the cubies unused in those specific cycles is not, so we can study where cubies go to perform specific moves on the cube.

7 Solving the cube

7.1 1st Layer

White Cross

First orientate the cube so that the yellow centre piece is facing upwards and white centre piece facing downwards, where the green face is our front face. Now match the white centre to the corresponding white edge pieces on the bottom.

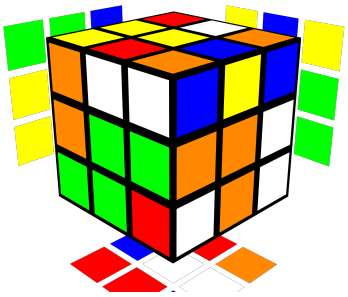

Figure 2: Scrambled position of Cube, with hint stickers to show the sides

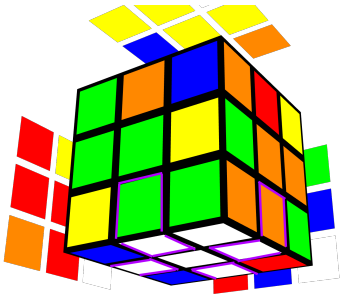

Figure 3: White edge pieces aligned forming white Cross (F'LBU2BF2)

White Corners

We will divide the cases into 2 types, the white corner facing towards the side and white corners facing upwards. For white corners facing towards the side, we align the corner piece next to the corresponding edge as shown in Figure 4. Then perform the commutator [U,R] which we call a right insertion.

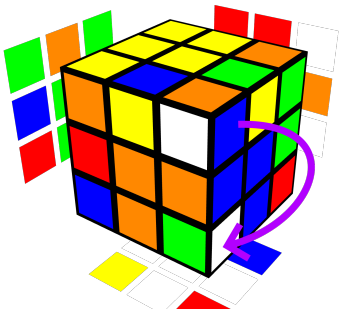

Figure 4: Right inserting white corner piece (F' U2 F y)

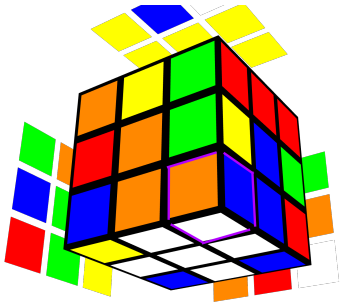

Figure 5: White corner piece right inserted ([U,R])

Left Mirror

Note: you may need to insert the corner piece into the left side instead of the right side, we can mirror our algorithm, by sending $U \mapsto U^{-1}$ and $R \mapsto L^{-1}$. The reason for the inversion is because R and L face in opposite directions so the clockwise rotation on both sides is different, and our corresponding left insertion is $[U]$.

For right corners facing upwards, we align the corner piece next to the corresponding edge as shown in Figure 6. Then we perform an insertion commutator perform the inverse insertion in the opposite direction, this twists the orientation of the corner without affecting any other pieces. [U,R] [F',U']. Note that the inverse of a commutator switches so we can think of [F',U'] as [U',L]. Make sure you are operating on with the white side facing towards you on the corner piece.

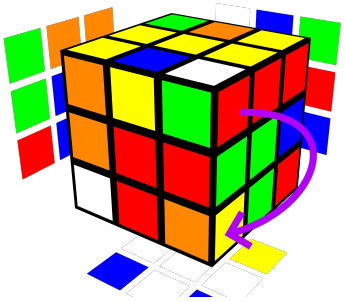

Figure 6: Right inserting white corner piece (F' U2 F y)

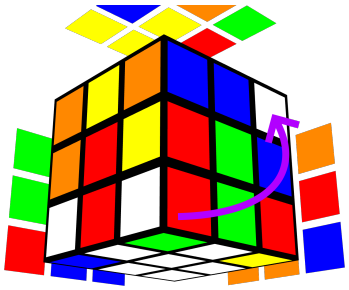

Figure 7

In figure 8 we can see that we just end up doing another insertion commutator once our corner has been flipped. We apply these rules to each corner until the entire bottom white side is finished.

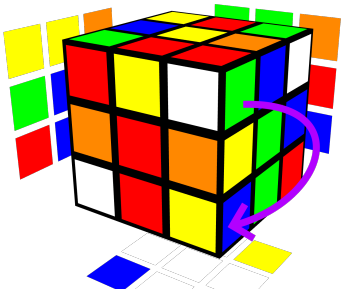

Figure 8

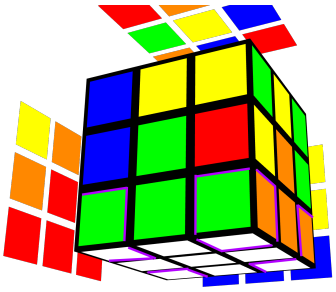

Figure 9

7.2 2nd Layer

This time we want to be able to change the orientation of our corner pieces. We do this by applying the commutator [U,R] or the left mirror [U',L'], however this time the commutator acts more like two conjugations. Consider [U',L']=$U^{-1}L^{-1}UL$ where our conjugations are $U^{-1}L^{-1}U$ and $L^{-1}UL$.

We start with the cube in the position with the corresponding side edge touching the starting side and the top piece telling us which direction to move the piece in.

$U^{-1}L^{-1}U$ Preserves the location of the centre edge piece but moves the left bottom corner piece to back left.

$L^{-1}UL$ Allows us to twist the left bottom corner clockwise.

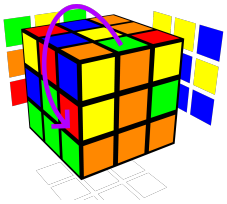

Figure 10

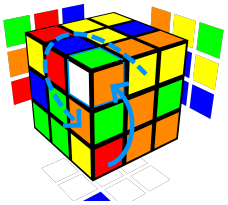

Figure 11

Figure 12

Now combing the last two figures we get figure 13. Repeating this for the right insertions for all the edge pieces we get the final figure 14.

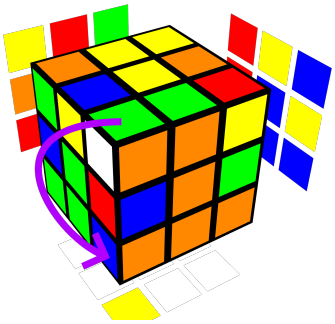

Figure 13: Right inserting white corner piece (F' U2 F y)

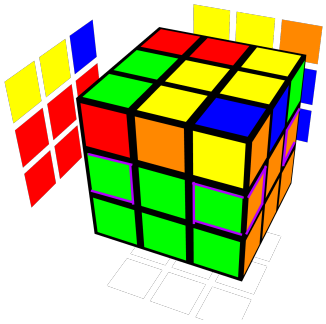

Figure 14

7.3 Final Layer

This is where our theory becomes more complicated and the 3-cycles we've developed become necessary to avoid mixing up the cube.

Yellow Cross

Consider our starting position in Figure 14, our first step is to solve for the yellow cross, we would like to use an edge 3-cycle which is our simplest move we can use to swap edge pieces. Our necessary condition to have a 3-cycle is that two basic moves must have a single cubie in common. This is quite natural as adjacent faces always have one cubie in common! So we can consider the commutator move [R,U] applied to Figure 14. To create a 3-cycle. We want to consider what cubies this move actually flips so we can create our 3-cycle by the following diagram:

We can see where our 3-cycle of our edge pieces go by working out the cycles. $R = (1\ 6\ 9\ 5)$ $U = (9\ 10\ 11\ 12)$ so $[R,U] = RUR^{-1}U^{-1} = (6\ 9\ 12)$

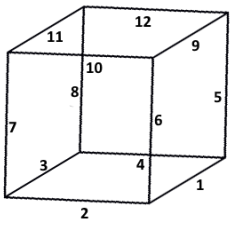

Figure 15: Right inserting white corner piece (F' U2 F y)

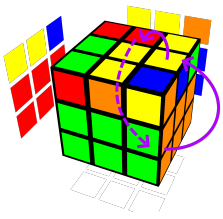

Figure 16: White corner piece right inserted ([U,R])

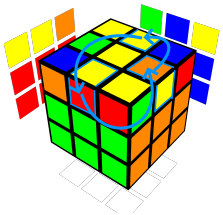

Figure 17

However we don't want to use cubie 6 as that will break our 2nd layer and we need to replace 9 with a cubie with a yellow edge on top. We can imagine by rotating our front face by F, cubie 10 would replace 6 in our starting position, but we don't want to break our white side so we perform a conjugation by F (which will not change our cycle structure) this now makes our 3-cycle (10 9 12).

So if we perform $F[R, U]F^{-1}$ on Figure 14 we get Figure 17

what do we do now? What we notice is that edge pieces 10 and 12 flip their orientation when given to the next piece, so if we move the side pieces 9 and 11 to 10 and 12 we can perform another side orientation swap and have all yellow side pieces facing upwards. $U = (9\ 10\ 11\ 12)$ maps 9 to 10 and 10 12 so performing U then $F[U, R]F^{-1}$.

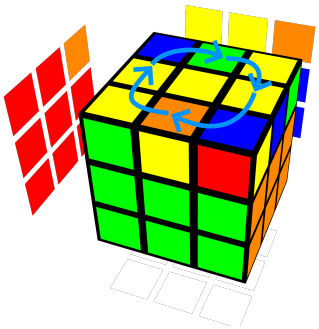

Figure 18

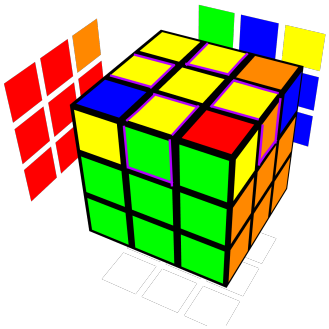

Figure 19

Orientating the Yellow corners

Suppose we have the position in Figure 19, we want to be able to twist all the yellow corners anticlockwise except the top left corner. we will use a basic commutator [R', U] to see where the corners end up and in which orientation. Using the notation in Figure 20 we get [R', U]=(1 8)(5 6) where cubies $1 \mapsto 8$ is a clockwise corner flip and $6 \mapsto 5$ is an anticlockwise corner flip.

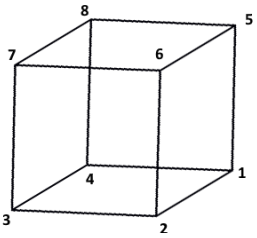

Figure 20: Right inserting white corner piece (F' U2 F y)

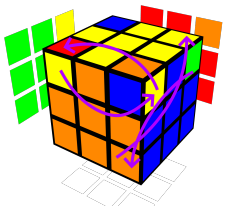

Figure 21

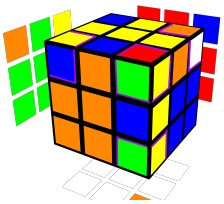

Figure 22

We want to see if we can remove cubie 1 from our cycle the simplest way to do this is to map 5 to 1 the means conjugating by R, however doing this maps 2

to 6 undoing our moves. So we want to move our top layer to allow 2 map to 7 which isn't any of our cubies we want to move. This requires the conjugating by RU. RU[R', U]U'R' which has the cycle structure (5 7)(8 6) it turns out 6 mapping to itself causes a clockwise corner flip. So as $6 \mapsto 8 = 6 \mapsto 6 \mapsto 5 \mapsto 8$ a clockwise corner flip and $8 \mapsto 6 = 8 \mapsto 5 \mapsto 6 \mapsto 6$/we have 2 anticlockwise corner flips which is a clockwise corner flip. It turns out every corner flip is now clockwise except $(7 \mapsto 5)$.

This is okay as In figure 19 we can perform U' so that the bottom top corner maps to 7 and as two anticlockwise rotations is a clockwise rotation we can just perform it twice making sure that 7 is the yellow that faces upwards.

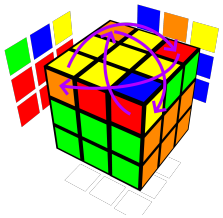

Figure 23

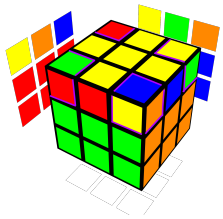

Figure 24: White corner piece right inserted ([U,R])

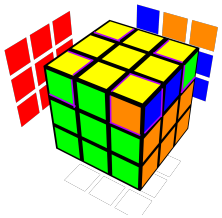

Figure 25

7.4 Orientating coloured sides of the last layer

Our first step is to orientate the corners, we'll consider using a commutator so we can 3-cycle our corners, we also want our to be more commutative to avoid edge flips, as we require to have a corner cubie contained in two moves to make a 3-cycle commutator we have the commutator $[F, R'B^2R]$ whos cycle composition is (2 8 6)

2 messes up our first layer so we need to replace it. The easiest way to remove to is to map 2 to 6 and 6 to 5 we can do this by conjugating by R' Thus our total composition is $R'[F, R'B^2R]R = (6\ 8\ 5)$

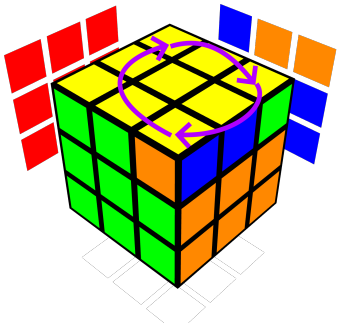

Figure 26

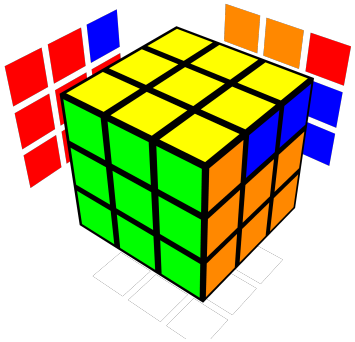

Figure 27

This didn't seem to fix the corner orientation as 5 did does not have the right coloured corners. Our problem is that we were cutting off our solved red corners in the cycle we need them to be in the cycle and for them both to map to something in the cycle such that their distance is preserved, the only way for this to be possible is for both of the solved red corners to be at the back, we can apply this to Figure 27 with the solved green corners by applying U2 first then $R'[F, R'B^2R]R$ then U just so our corners are aligned correctly.

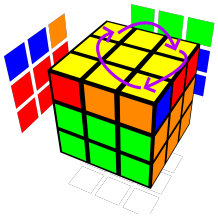

Figure 28

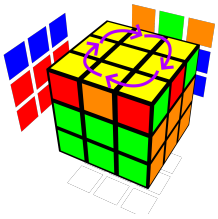

Figure 29

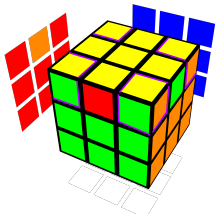

Figure 30

Now we need to find an edge 3-cycle which switches the coloured side, any two adjacent face moves are a 3-cycle edge commutator, specifically before we had an edge 3-cycle $RU[R', U]U'R'$ lets apply this to Figure 30.

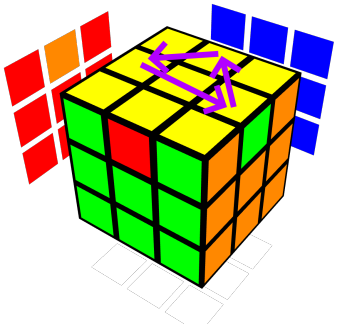

Figure 31: Right inserting white corner piece (F' U2 F y)

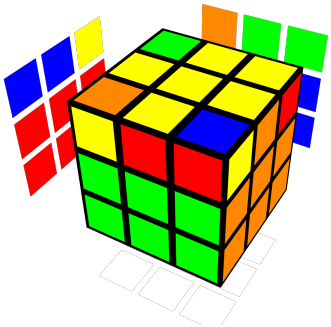

Figure 32

Unfortunately applying the commutator also twists the corners clockwise, however we have a trick if we perform a horizontal reflection of the commutator that will untwist the corners, but it also horizontally reflects our edge 3-cycle, so we require to have a finished piece at the back instead of the front, we see that the orange side is finished so if we conjugate our commutator by U' we can apply our edge 3-cycle and untwist our corners.

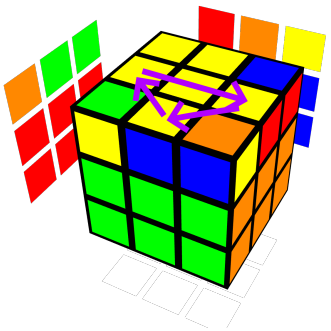

Figure 33:)

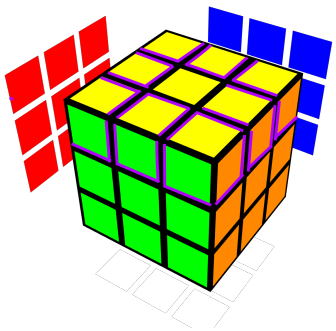

Figure 34

Animation of moves.

8 References

https://www.youtube.com/watch?v=mH0oCDa74tEt=187s

https://www.youtube.com/watch?v=EsBn7G2yhB8list=PLDcSwjT2BF$_V uNbn8HiHZKKy59SgnIAeO$

https://people.math.harvard.edu/ jjchen/docs/Group

https://dec41.user.srcf.net/notes/IA$_M$/*groups.pdf*

http://sporadic.stanford.edu/bump/match/rubik.pdf

https://web.mit.edu/sp.268/www/rubik.pdf

https://www.youtube.com/watch?v=0ob1m4XnVwQlist=PLDcSwjT2BF$_V uNbn8HiHZKKy59SgnIAeOin$
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https://math.stackexchange.com/questions/779546/can-rubiks-cube-be-solved-using-group-theory