Scratching the Surface of Infinity

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Abstract

In this essay, we explore how the notion of infinity has evolved over time and how it was foundational to the most important branches of mathematics. The earliest recording of infinity was in ancient Greece, where the concept of the physical existence of infinity was famously rejected due to the many paradoxes it would cause. After two millennia, a resolution to the paradoxes was found with the formal introduction of limits by Cauchy. In this essay we will investigate how the infinitesimal was quintessential to the foundations of Calculus, the different types of infinities proven by Georg Cantor and briefly cover the importance of the axiom of infinity in set theory. We hope that this will provide an understanding of infinity and how it is important in modern mathematics.

The Paradoxes of Infinity

It is famously known that the ancient Greeks rejected irrational numbers, such as $\sqrt{2}$ the hypotenuse of the unit right-angled triangle. As the ancient Greeks used geometric objects to aid their understanding of numbers, a side length that contained an infinite number of decimal places seemed nonsensical. Similarly, they also rejected the concept of infinity due to the seemingly impossibility of a physical infinity, whenever an attempt was made to define one it always resulted in a paradox. Two paradoxes that illustrate this are the stories of Zeno of Elea (Dichotomy Paradox) [1] and Achilles and the Tortoise. [2]

The Dichotomy Paradox:

- *Suppose that Zeno wanted to walk to the park.*
- *The distance between his house and the park is of finite length.*
- *To get to the park, Zeno must first walk halfway to the park.*
- *And to get halfway to the park, Zeno must first walk one quarter of the way to the park.*
- *And to get one quarter of the way to the park, Zeno must walk one eight of the way.*

Etc.

Achilles and the Tortoise:

Achilles races a tortoise, giving the latter a head start.

- *Step #1: Achilles runs to the tortoise's starting point while the tortoise walks forward.*
- *Step #2: Achilles advances to where the tortoise was at the end of Step #1 while the tortoise goes yet further.*
- *Step #3: Achilles advances to where the tortoise was at the end of Step #2 while the tortoise goes yet further.*

- *Step #4: Achilles advances to where the tortoise was at the end of Step #3 while the tortoise goes yet further.*

Etc.

Zeno of Elea had split up the finite distances into infinitely smaller distances. What this meant was that there was no first distance to travel as it could be split into a smaller distance, hence the journey could not even begin. His paradoxical conclusion led to the belief that motion was an illusion. While Zeno of Elea was not attempting to make a point about infinity, the paradox shows how counterintuitive infinity can be. This paradox was just a foreshadow of our modern understanding that we can do something infinitely many times and obtain a finite result. This resulted in Aristotle distinguishing between potential infinity and actual infinity. Potential infinity is the idea of an infinite process, whereas actual infinity is that something is actually infinitely long. The reason for the distinction was because they said that infinity could never happen in the real world.

Modern Day Infinity

In the 17th century, European mathematicians started using infinite numbers and infinite expressions in a rigorous fashion. The first to do so was John Wallis who, in 1655, used this new notation for such a number in his *De sectionibus conicis*. [3] He went on to use the concept of infinity to calculate area by dividing the region into infinitesimal rectangles of width on the order of $\frac{1}{\infty}$. But in Arithmetica infinitorum (1656), his notation changes and he indicates infinite series, infinite products, and infinite continued fractions by writing down a few terms or factors and then appending "&c.", as in "2, 4, 6, 8, 10, &c." This could be seen as a precursor to the development of calculus and John Wallis was given partial credit for its development. [4]

Infinite series

To resolve the paradoxes brought up earlier we need to look at infinite series. However I would first like to define some mathematical terms to clarify my explanations:

- A set is a collection of objects written using curly brackets for example, {1,2,4,3} is a set and {3,4,1,2} is the same set because order does not matter.
- A sequence is an ordered set, written without the curly brackets. The sequence 1,2,3,4 is a different sequence to 3,2,1,4 because order matters.
- A series is the sum of the terms in a sequence, for example 1+2+3+4 is a series and so is 3+2+1+4.

There are different types of sequences:

An arithmetic sequences is when the terms of a sequence differ by a common difference,
$a,\ (a + d),\ (a + 2d),\ (a + 3d),$
Where a is the first term and d is the common difference,

For example:
1, 3, 5, 7, 9,
The nth term of this sequence is $a + (n - 1)d$.

A geometric sequence is when the terms of a sequence differ by a common ratio,
$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ ar^5,$
Where a is the first term and r is the common ratio,
For example:
5, 50, 500, 5000, 50000,
The nth term of this sequence is ar^{n-1}.

A harmonic sequence is when the reciprocals of the terms follows an arithmetic progression,
$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \frac{1}{a+3d}, \frac{1}{a+4d},$
Where a is the reciprocal of the first term and d is the difference of the reciprocals,
For example:
$\frac{1}{5}, \frac{1}{10}, \frac{1}{15}, \frac{1}{20}, \frac{1}{25},$
The reciprocal of the terms in this sequence is,
5, 10, 15, 20, 25,
Which follows an arithmetic progression,
The nth term of this sequence is $a_n = \frac{1}{a + (n-1)d}$.

A recursive sequence is a sequence that is defined based one or more of its previous terms,
For example, the fibonacci sequence:
$a_1 = 1,$
$a_2 = 1,$
$a_n = a_{n-1} + a_{n-2},$
$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13.$

An infinite sequence is a sequence which has an infinitely many terms.
They can be convergent or divergent. If an infinite series is convergent then its sum is a finite number, whereas divergent series have infinite sums.

In 1821, Augustin-Louis Cauchy had developed a formal definition of convergence for an infinite geometric series where the common ratio is between 0 and 1. In Cauchy's *Cours d'Analyse de l'École Royale Polytechnique*, page 124. [5]

For 0 < r < 1,
$a + ar + ar^2 + ar^3 + ar^4 + ar^5 + \ldots = \frac{a}{1-r}$.

So how did Cauchy arrive at this?
Let us look at some geometric series.
Example 1:
$1 + 10 + 100 + 1000 + 10000 + ... = +\infty$
In this geometric series the first term is 1 and the common ratio is 10 so we can say this would equal positive infinity.

Example 2:
What about this series?
$0.9 + 0.09 + 0.009 + 0.0009 + ... = S$

If we assume this geometric series is convergent, then we can say it is equal to some finite number, S. So how can we go calculate this finite number? Looking at the first few terms of this series we get 0.9999, what this suggests is that this infinite sum gets closer to 1. But does it equal to 1? To determine whether this is the case we need to learn some rules that must be followed when dealing with infinite series.

1. You are allowed to shift an infinite series,

You can think of this as adding a zero in front of the series.
$1 + 2 + 3 + 4 + 5 + ... = +\infty$
$0 + 1 + 0 + 0 + 2 + 3 + ... = +\infty$

2. Term by term addition and subtraction,

to add to convergent series you can line them up like this:
$a_1 + a_2 + a_3 + a_4 + a_5 + ... = S_1$
$b_1 + b_2 + b_3 + b_4 + b_5 + ... = S_2$
$(a_1 + b_1) + (a_2 + b_2) + (a_3 + b_3) + (a_4 + b_4) + ... = S_1 + S_2$

3. Multiplication and division

$a_1 + a_2 + a_3 + a_4 + a_5 + ... = S_1$
$r\,(a_1 + a_2 + a_3 + a_4 + a_5 + ...) = r(S_1)$

How can we apply this to our second example:
$0.9 + 0.09 + 0.009 + 0.0009 + ... = S$
In this geometric series, the first term is 0.9 and the common ratio is 0.1, if we multiply both sides of the equation by 0.1 we obtain
$0.09 + 0.009 + 0.0009 + 0.00009 + ... = 0.1S$
Next we shift it to the right by 1 term
$\quad\quad 0.09 + 0.009 + 0.0009 + 0.00009 + ... = 0.1S$
Subtracting S from 0.1S by term wise subtraction, we obtain
$0.9 + 0.09 + 0.009 + 0.0009 + ... \quad\quad = S$

$$0.09 + 0.009 + 0.0009 + 0.00009 + \ldots = 0.1S$$

All subsequent terms would cancel, giving us

$$0.9 = 0.9S$$

Dividing by 0.9, we obtain that S = 1, suggesting that the series has a finite value of 1. Namely that 0.9999… is equivalent to 1.

Using the same algebraic steps for the general geometric sequence we obtain that,
For 0 < r < 1,

$$a + ar + ar^2 + ar^3 + ar^4 + ar^5 + \ldots = \frac{a}{1-r}$$

Because if r was greater than 1, the series would diverge (refer to example 1).

Defining a Limit

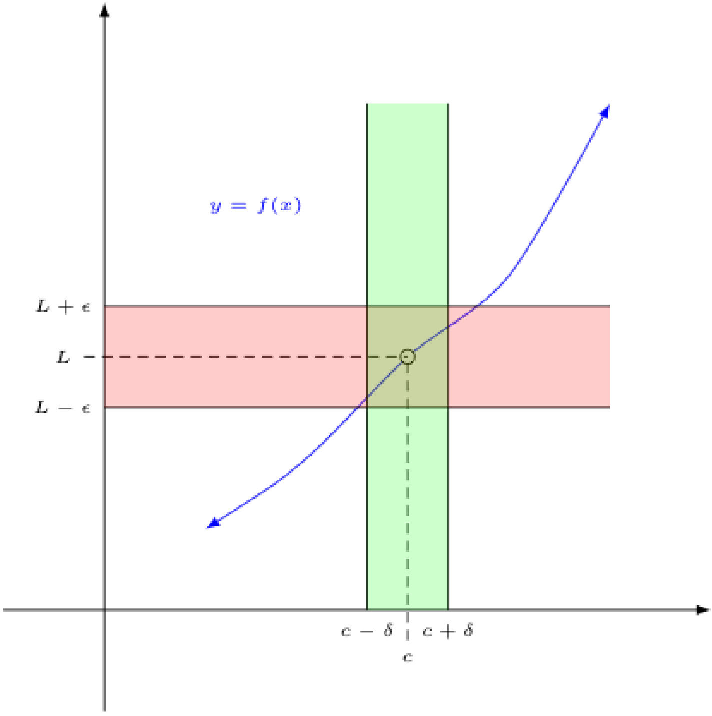
Previously, we have made the assumption that the infinite series was convergent in order to manipulate it. So how can we tell if an infinite series is divergent or convergent? One mathematician who pioneered convergence tests was Cauchy; to do this, he had to first define what a limit was. After Cauchy had first defined a limit, this was later finalised by Weierstrass in 1861.

Formally speaking, the definition of a limit is the (ε, δ)-definition of limit:

$$\lim_{x \to c} f(x) = L \;\; means$$

$$(\forall \varepsilon > 0),\ (\exists \delta > 0),\ (\forall x \in \mathbb{R})\ such\ that\ (0 < |x - c| < \delta \Rightarrow |f(x) - L| < \varepsilon)$$

The technical definition is something that we will no go in depth on but it essentially means that 'for every real $\varepsilon > 0$, there exists a real $\delta > 0$ such that for all real x, $0 < |x - c| < \delta$ implies $|f(x) - L| < \varepsilon$.' [6]



(figure 1.)

Using figure 1 to help us visualise, the formal definition says that for any distance on the y axis , the red boundary, there is another distance on the x axis, the green boundary, so that for every value of x, that is with the green boundary, that is also with the red boundary.

It means that as the value of x approaches the point c from both sides, which in this case is undefined, there is a value that y gets closer to.

For example, given the function f(x), and f(c) is undefined. Then we can take $f(x + \delta)$ and $f(x - \delta)$ and see what value both function approaches. For example if c = 2, then we take f(2 ± 0.1) and make δ smaller, f(2 ± 0.001). Etc. Then the value this converges on is equal to L. The limit can be interpreted as the f(c ± δ) and the value they approaches, using our example then

$\lim_{x \to 2} = L$.

In some cases there may not be a limit.

The idea of a limit, avoids the use of infinity and infinitesimals. The idea of infinity, as discussed in the paradoxes, cannot happen in real life. The use of limits is the idea that an infinite series cannot be greater than a finite number, if it is an increasing function; or less than a finite number, if it is a decreasing function. By a increasing function I mean an infinite series whose sum increase with the number of terms going to infinity, for example,

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + ... = 2$$

Whereas a decreasing function, is an infinite series whose sun decreases with the number of terms going to infinity, for example,

$$-1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} - \frac{1}{16} - ... = -2$$

Divergence Test

Now that we have defined what a limit is, we can use it to test if an infinite series is divergent or convergent.

For an infinite series if the terms do not approach zero
$for\ an\ infinite\ \ sequence\ a_n.$

$if\ \ \lim_{n\to\infty} a_n \neq 0$, then we can say the infinite series is divergent. [7]

This means that if the terms of an infinite series do not approach zero, then the series is divergent.

What does this mean? For an infinite series like,
1 + 10 + 100 + 1000 +...

We can see that as the nth term increases the value of the nth term increases to infinity. This means it is divergent.

However if,
$\lim_{n\to\infty} a_n = 0$, then the test is inconclusive.

For example, the harmonic series is actually divergent
$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}...$

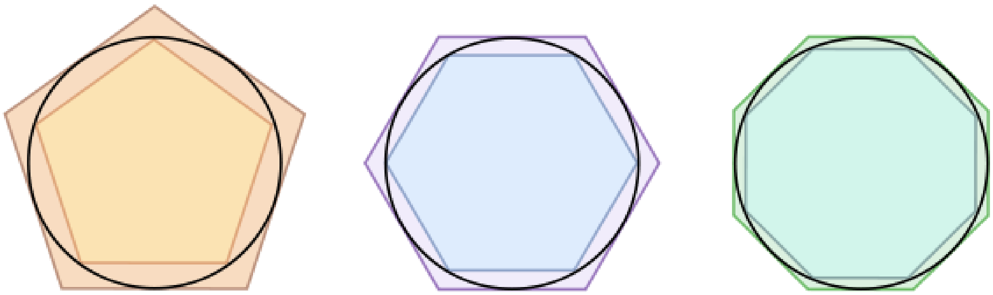
Where as
$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} +... \ = 2$

Looking at the terms of both sequences we can see they both get smaller and smaller and they approach zero. However, this does not necessarily mean that the sum of the sequence would be finite.

Calculus

Calculus is the study of continuous change. It has two major branches called differential calculus and integral calculus. Differential calculus studies the ‘instantaneous’ rates of change while integral calculus is concerned with the area under curved bodies.
The modern notion of calculus was independently founded by Isacc Newton and Gottfried Wilhelm Leibniz. [8]

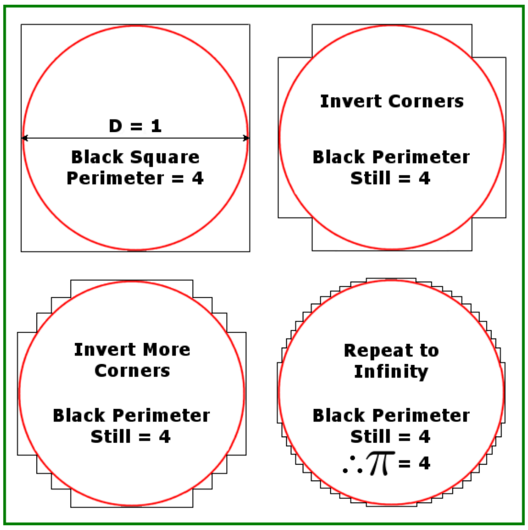
How does calculus work?
In ancient Greece, the method of exhaustion was made rigorous by Eudoxus of Cnidus. Archimedes used it to calculate the area of a circle, by inscribing polygons with an increasing number of sides into the circle. [9]



(figure 2.)

In figure 2 we can see the area of the circle is bigger than the shape inscribed by the circle and smaller than the similar shape outside the circle. By increasing the number of sides the shape has, from 5 sides in the first diagram to 6 sides in the second, the limit of the area would get closer to the circle.

However, the area would never be identical to the circle unless the polygon had infinitely many sides. The idea of increasing the number of sides of the polygon, could also be seen as decreasing the side length of the polygon, in the second diagram we see a hexagon with a side length that is smaller than the pentagon in the first diagram. The idea of decreasing the side length to approximate the area is how calculus works.
So how is the idea of decreasing the side lengths useful?

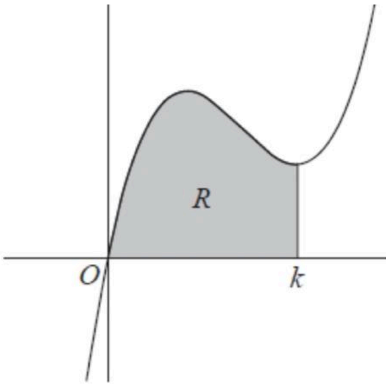


(figure 3.)

If we didn't decrease side lengths then this could lead to some contradictions, for example you could construct a ‘proof’ that $\pi = 4$. As seen in figure 3
The problem with this proof lies in the second step. The second step does not decrease the side length of the square. Meaning if we continue this process indefinitely it will not be a smooth circle, but have jagged edges. To properly use infinity to approximate curves you can do some further research by watching this video. [10]

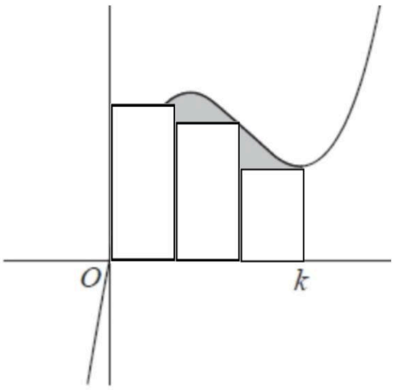
Integration

Integral calculus, deals with calculating the area under a curve. It can be thought of as having infinitely many rectangles with a width approaching zero.
How can we find the area R in figure 4?
First we can draw some rectangles of equal width.

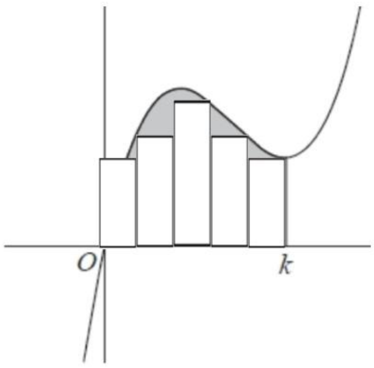


(figure 4.)

In figure 5 there are three rectangles of equal width. Just like archimedes increasing the number of sides in the method of exhaustion, we can do the same thing if we increase the number of rectangles, as seen in figure 6.



(figure 5.)



(figure 6.)

This decreases the width of the individual rectangles as their total width must be equal to k. Therefore as the width decreases and the number of rectangles increases to infinity, the sum of the area of the rectangles becomes equal to the area under the curve. And when there are infinitely many rectangles then we get the idea that the rectangles have a width of $\frac{1}{\infty}$. This is the fundamental principle of integration. However the idea of a rectangle having an infinitely small width can be somewhat paradoxical. So we use the idea of limits. The area under the curve is equal to the area of the rectangles as the number of rectangles tends to infinity.

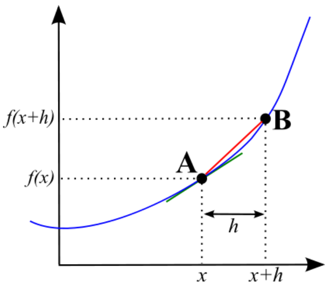
The formal definition of integration is:

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{n=1}^{n} f(x_n)\Delta x$$ [11]

This means, the area under the curve in the interval [a,b] is equal to the sum of rectangles under the curve as their width gets smaller because we increase the number of rectangles.

Differentiation

Differential Calculus also uses infinity but to calculate the rate of change.
The idea of decreasing a distance such that it closely approximates the area used in integration can be applied here.
If we look at figure 7, then we can see that point B is a distance of h away from point A. If we decrease h then the value of the gradient of the line segment AB is going to approach the instantaneous gradient at A. This uses the idea of limits. When differentiation was first it did not have any theoretical standing. He used the idea of infinitesimals by considering point B to be an infinitely small distance away from point A. [12]



(figure 7.)

To calculate the gradient of the line segment we can use the change in x divided by the change in y. The point B has coordinates $(x + h, f(x + h))$ and the coordinates of A is $(x, f(x))$.
So the gradient of the line segment is $\frac{\Delta y}{\Delta x} = \frac{f(x+h)-f(x)}{(x+h)-(x)} = \frac{f(x+h)-f(x)}{h}$

To get the gradient of the line segment to approximate the gradient at A, we would want h to be as small as possible, so we say:

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$ [13]

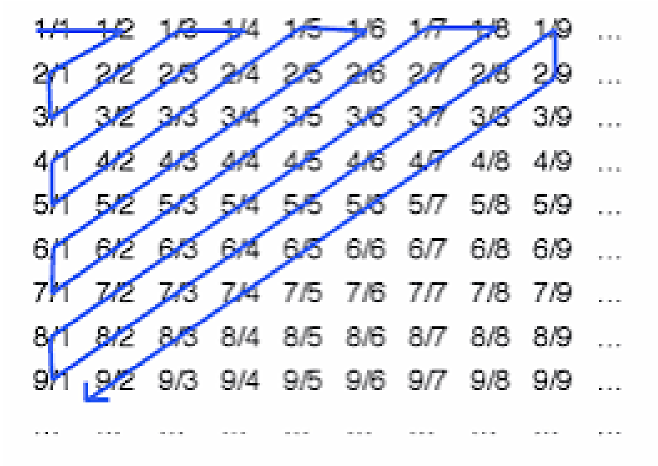
Which is the formal definition of differentiation.

What this means is that the gradient at the point A is equal to the gradient of the line segment AB as B gets closer to A.

Making sense of the uncountable

In Mathematics, infinity is used to denote a never-ending property. Surprisingly, there are two types of infinity: countable and uncountable. So what's the difference? A good analogy to explain this is the story of *"the Accident in Hell"*. Imagine that you were sent to hell accidentally and you plead to the devil to be sent to heaven. Eventually, a deal is made: the devil will think of any positive integer (the set of natural numbers) and if you guess it you are sent to heaven. So what strategy should you use? This is quite simple, if you start from the number one and count upwards, you would hit each integer given an extensive period of time. An infinite amount of time, but you'll eventually guess the number, because the number is a finite value. If the devil picks 10, then you will take 10 guesses, if he picks a google, then it'll take a google guess, but no matter how big the number is, you will still be able to guess correctly in a finite amount of time.

But what if the rules of the deal change? Now, the devil can pick any rational number. Can you still guess correctly? A rational number can be written as a fraction. A fraction is composed of a numerator and a denominator, $\frac{a}{b}$, where a and b are integers. To count every fraction let's consider the positive fractions first. Count every fraction where a and b sum to 1. In this case the only example is $\frac{0}{1}$. Here we can also count $-\frac{0}{1}$. Then we count the fractions where a and b sum to 2. So $\frac{0}{2}$, $-\frac{0}{2}$, $\frac{1}{1}$, $-\frac{1}{1}$. One more example, a and b sum to 3: $\frac{0}{3}, -\frac{0}{3}, \frac{1}{2}, -\frac{1}{2}, \frac{2}{1}, -\frac{2}{1}$. What this does is create a staircase pattern as seen in figure 8.



(figure 8.)

What if the game was changed yet again? This time, the devil can pick any irrational number, is there any way for you to win? The answer is no. There is no systematic way to list every irrational number, as there is no clear link between them; the numbers could have an infinitely long decimal or none at all. This means counting in order will not work, as if we start from zero there is no obvious step to get to the next number; there will always be a smaller step because irrational numbers can have an infinite amount of digits (such as e and Pi).
How do we prove this?

Cantor's Diagonalization Proof

In 1874 Georg Cantor [14] published "On a Property of the Collection of All Real Algebraic Numbers" which proved that in any interval [a,b] there are infinitely many transcendental numbers. The first part of his theorem was proved using the diagonalisation argument. This is a proof by contradiction.
You start by assuming you write a complete and infinitely long list of all the irrational numbers, as seen in figure 9.

To construct a new irrational number we take the first digit of the first irrational number, this will be the first digit of our new irrational number.

Take the second digit of the second irrational and add one, this will be the second digit. This will carry on forever and we have constructed another irrational number that is not present in our table, because it is different from every irrational number by at least one digit.

This contradicts our assumption that we have written down every irrational number. Therefore it is impossible to write down every irrational number as we will always miss out on some irrational numbers.

$$
\begin{array}{ll}
s_1 & = 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0 \ldots \\
s_2 & = 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \ldots \\
s_3 & = 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0 \ldots \\
s_4 & = 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1\ 0\ 1 \ldots \\
s_5 & = 1\ 1\ 0\ 1\ 0\ 1\ 1\ 0\ 1\ 0\ 1 \ldots \\
s_6 & = 0\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0 \ldots \\
s_7 & = 1\ 0\ 0\ 0\ 1\ 0\ 0\ 0\ 1\ 0\ 0 \ldots \\
s_8 & = 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1 \ldots \\
s_9 & = 1\ 1\ 0\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 0 \ldots \\
s_{10} & = 1\ 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1\ 0\ 1 \ldots \\
s_{11} & = 1\ 1\ 0\ 1\ 0\ 1\ 0\ 0\ 1\ 0\ 0 \ldots \\
\vdots & \quad \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \vdots\ \ddots
\end{array}
$$

$$
s = 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0\ 0\ 1\ 1 \ldots
$$

(figure 9.)

With his paper published in 1874, Georg Cantor is often called the founder of set theory. Although his set theory was often called 'naive set theory' due to the fact that it allowed paradoxes. Now modern set theory, called Zermelo Fraenkel Set theory, contains 8 axioms, of which, the axiom of infinity guarantees the existence of an infinite set. This is important because it allows us to encode the natural numbers as a successor of the previous number. Where zero is encoded as the empty set. This will not be delved into much. However the axiom of infinity is important because the other axioms, if consistent, are unable to prove the existence of the infinite set. [15]Set theory is important, because it covers almost all of maths.

Conclusion

In conclusion, we have explored how the concept of infinity has evolved over time. The ancient Greeks had separated potential infinity from actual infinity, while showing the paradoxical nature of infinity. Then in 1655, briefly covered the first appearance of '∞', and how he used an informal definition of infinity to calculate area. After two millennia, with the formal definition of limits, Augustin-Louis Cauchy had resolved the paradoxes, showing that an infinite series can have a finite limit. We investigated how the ancient Greeks used the idea of a potential infinity to calculate the area of a circle, and how one has to be careful when using infinity to approximate curves. Then we went onto how Newton and Leibniz could have developed calculus using the idea of infinitesimals and the formalisation of calculus with the introduction of limits. Finally, we show how Cantor proved that there were different sizes of infinity and a way to understand them intuitively and how infinity is important in set theory.

How has the concept of infinity changed? At first it was something completely paradoxical as infinity could never be achieved in the real world, like adding all the terms in an infinite sequence. However, with the idea of limits it removes the need to consider an infinite amount of rectangles when calculating an integral, instead the use of limits allows us to set bounds for which the infinite series will never be greater than.

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