Stepping Into New Dimensions, Shay Nagda, March 2024

Introduction

Visualisation is an important step in the process of human understanding, particularly in mathematics. It is, after all, the reason why number lines and word problems are used to teach the basic concepts of maths to children, and the reason why graphs can be such powerful tools to mathematicians. However, when dealing in more than three dimensions, visualisation becomes very difficult, very quickly. In the same way that a cartoon character could not possibly conceive of a shape that extends outwards from the screen, we as three-dimensional beings struggle to visualise shapes in four dimensions (and find it virtually impossible in five or more). Yet, despite this obstacle, we can still, using our limited understanding of dimensionality, construct patterns that allow us to predict and imagine the overall properties of multi-dimensional shapes. This is exactly what I shall aim to explore, looking specifically at hypercubes (the n-dimensional forms of a cube).

1. Composition

There exists a (surprisingly elegant) formula to find the elements and composition of an n-dimensional hypercube (notated as an 'n-cube'), and in this essay I will attempt to explain, as intuitively as possible, where this formula comes from and why it works. But what do we mean by the 'composition' of a hypercube? Strictly speaking, shapes in more than three dimensions do not exist to us...so how do we define them?

To start to understand this we first need to know, fundamentally, what we are really doing when we move from one dimension to the next. We can use the simple examples we already know – points, lines, squares, and cubes (which are hypercubes of 0, 1, 2, and 3 dimensions respectively).

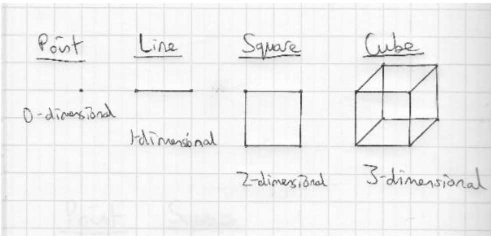


Figure 1: 0,1,2 and 3 dimensional cubes

From these examples, we can create a rule to go from an n-cube to an $(n+1)$-cube: the shape is extended out along the new dimension, by the same distance as all previous edges (hypercubes are regular). As for what we mean by the 'new dimension', the (informal) mathematical definition of the number of dimensions in a shape is the minimum number of coordinates needed to express any of its points[1]. In other words, each dimension is completely independent of the others, and movement in its direction cannot be observed through any of the other dimensions.

Therefore, we can say that by extending the shape in the new dimension, its pre-existing properties in each dimension remain unchanged, and are merely extruded outwards. Since this is the case, we can observe that the forming of an n-cube is essentially the joining of two $(n-1)$-cubes to form a 'prism' with an $(n-1)$-dimensional cross section:

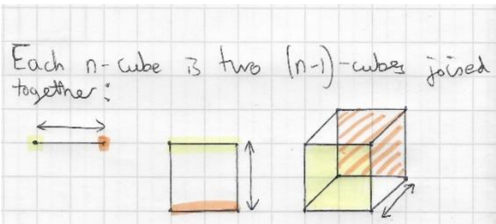


Figure 2: Forming an n-cube

[1] Definition source: https://en.wikipedia.org/wiki/Dimension

These $(n-1)$-cubes are in turn composed of $(n-2)$-cubes...and so on. Therefore, an n-cube is composed of a combination of all lesser n-cubes. This can be observed in the examples in *Figure 1* – a line is composed of 2 joined points, a square is composed of 4 lines (edges) and 4 points (vertices), and a cube is composed of 8 vertices, 12 edges and 6 squares (faces).

Using this joining idea, we can form a representation of a 4-cube, known as a tesseract:

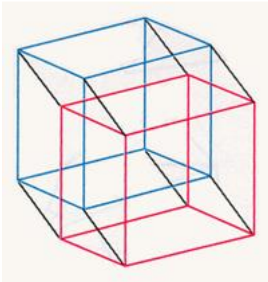


Figure 3: A Tesseract [2]

This provides a useful alternative to visualisation – although we cannot fully comprehend hypercubes, we can express their properties by finding their formation as a combination of elements, expressed succinctly as a formula for the number of k-cubes present in an n-cube (where k is the dimension of the element in question, and n is the dimension of the overall hypercube).

2. How many corners in a square?

To reach the final equation, we can begin by creating a formula for the simplest of k-cubes i.e. 0-cubes (vertices). We can then use the number of vertices in an n-cube to find the number of edges (1-cubes) and then use this to find the number of faces (2-cubes), and so on.

We have already found that, in order to create an n-cube, we must extend an $(n-1)$-cube. Since this extension is uniform, the only additional vertices that are created are those that form the two end boundaries. As a result, the number of vertices doubles. An alternative way of thinking about this is that the creation of an n-cube is the joining of two $(n-1)$-cubes, thus doubling the number of vertices present.

Therefore, we can say that the number of vertices doubles with each dimension of a hypercube, starting at 1 vertex in a 0-cube (a 0-cube is a single point i.e. a vertex). So we can express the number of vertices in an n-cube simply as 2^n . We can check this with our previous examples: a 1-cube is a line segment with two vertices, a 2-cube is a square with four vertices, and a 3-cube is a cube with eight vertices. Using the image of the tesseract, we can also see that a 4-cube has 16 vertices.

However, the complication arises when considering the next k-cube, being an edge (a 1-cube). This time, the process of extending the shape, as well as doubling the number of edges, also creates new ones along the extrusion – meaning the method we used for vertices does not apply. However, we already have the number of vertices in an n-cube, which can be used to figure out the number of edges in the cube. We can do this because we know that the number of vertices joined by each edge of a hypercube is 2 (this is the very definition of an edge), no matter how many dimensions we are in. But this is not the only piece of information we need – not only does each edge join two vertices, but each vertex joins a certain number of edges. As we can see below, this ratio is not the same in any dimension:

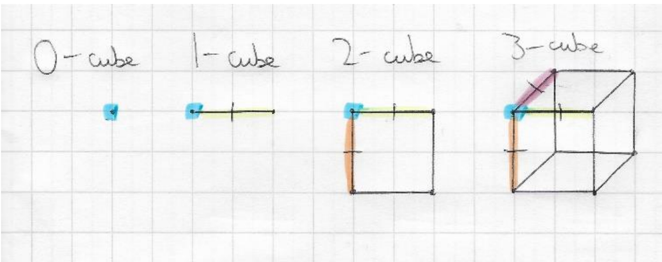


Figure 4: Number of edges meeting at a vertex in 0,1,2 and 3 dimensions

[2] Image source: *https://www.math.brown.edu/tbanchof/Beyond3d/chapter4/section07.html*

In a 1-cube, each vertex is adjacent to one edge; in a 2-cube, each vertex joins 2 edges; in a 3-cube, each vertex joins 3 edges; and in a 4-cube, each vertex joins 4 edges. An obvious pattern is emerging here, but why is this the case? The answer is fairly intuitive – we've seen that each dimension is an extension of the 0-cube, a vertex, in a new direction, thus creating a new line for each dimension. Therefore the number of dimensions is equal to the number of edges meeting at each vertex. To create a formula for the number of edges in an n-cube, we can take the number of vertices (2^n) and multiply by the number of edges that meet at each one, which we have found to be n. However, this would be double-counting, since each of those edges will always join two vertices, creating an overlap. To counter this, we will halve the answer, giving:

$$\frac{n2^n}{2}$$

Which simplifies to:

$$n2^{n-1}$$

To achieve the number of 2-cubes (faces), we can thus use a very similar process, taking the number of edges in an n-cube, multiplying by the number of faces that meet at each edge, and dividing by the number of edges on each face, which remains constant at 4 edges per square face. For the same reason that the number of edges meeting at a vertex increases by one for each new dimension introduced, the number of faces meeting at each edge will also increase by one for each dimension – however, it will not be equal to n again, since faces are only present in $(n \geq 2)$- cubes (the number of faces meeting at an edge in a 2-cube i.e. square, is 1 since the entire 2-cube is itself a face). Therefore, while the number of edges meeting at a vertex was n, the number of faces meeting at an edge is $n-1$. Checking against the examples in *Figure 1* that the number of faces meeting at an edge is 0 in a 1-cube, 1 in a 2-cube, 2 in a 3-cube, and 3 in a 4-cube:

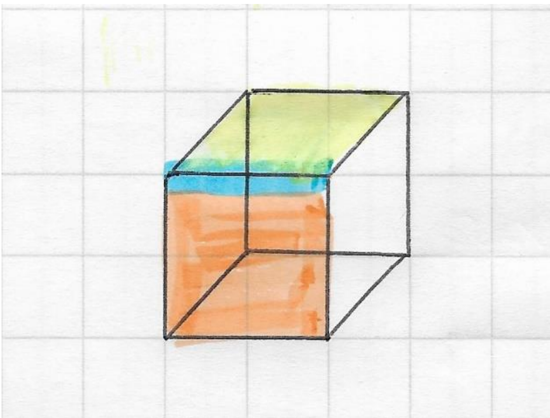


Figure 5: Number of faces meeting at an edge illustrated in a 3-cube

In this way we can construct another formula, this time to find the number of faces in an n-cube. In similar fashion to the formula for finding the number of vertices, we will multiply the number of edges by the number of faces meeting at each, this time dividing by 4, since there are 4 edges to each face:

$$(n-1)\frac{n2^{n-1}}{4}$$

Simplifying to:

$$(n-1)(n2^{n-3})$$

3. Generalisation

The method we have used to construct the specific formulae for edges and vertices, which is fundamentally the same, can logically be applied for any element of a hypercube, to give a more general formula for k-cubes within an n-cube. Each time, we have based the specific formulae on three values:

- the number of (k-1) cubes present in the given n-cube
- the number of (k-1) cubes contained within each k-cube
- the number of k-cubes which meet at each (k-1) cube in the given n-cube

This gives us a general formula as follows:

$$(\boldsymbol{number\ of\ k - cubes\ meeting\ at\ each\ (k-1) - cube\ in\ the\ given\ n - cube})\frac{\boldsymbol{number\ of\ (k-1) - cubes\ in\ the\ n - cube}}{\boldsymbol{number\ of\ (k-1) - cubes\ in\ the\ k - cube}}$$

Now, in order to arrive at a more useful formula, we need to consider each of these three parts, just as we have already done for edges and faces:

Firstly, we can consider the number of $(k-1)$-cubes contained within each k-cube (the divisor in the general formula). This does not depend on n, since k-cubes will always be the same no matter what structure they lie within (a square is a square whether it's the face of a cube, a tesseract, or anything else). This is quite simple, since we already know that a k-cube, by definition, has $(k-1)$-cubes as its boundaries. These boundaries form the start and end points of the shape in each direction. And since the number of directions is determined by the number of dimensions, which is k, we know that the number of boundaries of the shape must be 2k. Therefore, the number of $(k-1)$-cubes in a k-cube is given by $2k$. The image below illustrates this using the examples in *Figure 1*:

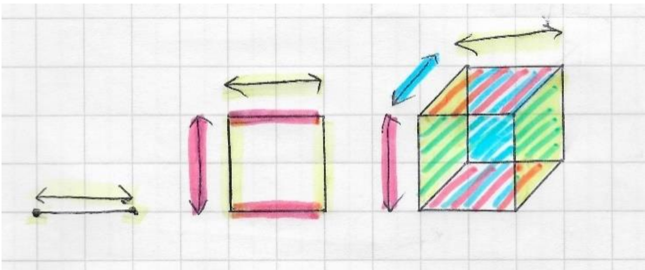


Figure 6: Each k-cube has two (k-1)-cube boundaries for each dimension

Now we can consider the multiplier of the formula, the number of k-cubes meeting at each $(k-1)$-cube, which we can find by extending the logic used to find the number of edges meeting at a vertex in an n-cube, and to find the number of faces meeting at an edge in an n-cube. As we saw when looking at faces, we know that:

- The number of k-cubes meeting at a $(k-1)$-cube increases with n
- There are only k-cubes present when $n \geq k$, and therefore they are not present when $n \leq k-1$

Therefore, we can say that the number of k-cubes meeting at a $(k-1)$-cube would be equal to n, but we must exclude $k-1$, thus giving $n-(k-1)$ or $n-k+1$:

	Edges at each vertex ($k=1$)	***Faces at each edge ($k=2$)***	***Cells at each face ($k=3$)***
Line ($n=1$)	*1*	*n/a*	*n/a*
Square ($n=2$)	*2*	*1*	*n/a*
Cube ($n=3$)	*3*	*2*	*1*

At this point, we have generalised the formula to:

$$(\boldsymbol{n}-\boldsymbol{k}+\mathbf{1})\frac{\boldsymbol{number\ of\ (k-1)-cubes\ in\ the\ n-cube}}{\mathbf{2}\boldsymbol{k}}$$

4. Recursion

This formula is easier to use than the first general formula, but we can certainly go further. We can see that this formula is essentially a recursive relation – it gives the answer for k based on the answer for $k-1$. Of course, this is not practical to use as a formula, so we need to remove this recurrence to create a single, usable formula. To do this, we can begin with the number of 0-cubes (vertices) in the n -cube, and put this number through the formula k times. We have already found the number of vertices in an n-cube to be 2^n, and since the formula consists of a divisor and a multiplier, we can approach each separately, conducting them k times.

Firstly, we'll approach the divisor, $2k$. We cannot simply divide by $(2k)^k$ instead, since the value of k changes with each iteration of the formula. It will start at 1 as we find the number of edges present, then become 2 for faces, then 3, and so on until it reaches k. Therefore, the process will be:

$$(2\times 1)(2\times 2)(2\times 3)\dots(2\times k)$$

Since the number of brackets is k, the twos at the start of each bracket will simplify to 2^k, giving:

$$2^k(1\times 2\times 3\dots\times k)$$

We can now see that the second bracket is simply $k!$, meaning the divisor, corrected for recursion, will be:

$$2^k k!$$

Next we can approach the multiplier, $n-k+1$. As we saw previously, k will change with each iteration of the formula, starting at 1 and going up to k. However, n, which is the number of dimensions of the overall hypercube, will not change. Therefore, we can write this sequence as:

$$(n-1+1)(n-2+1)(n-3+1)\dots(n-k+1)$$

This simplifies to:

$$(n)(n-1)(n-2)\dots(n-k+1)$$

To express this using factorials, we can consider $n!$, which we could represent as:

$$(n)(n-1)(n-2)\dots(n-k+1)\dots(1)$$

Since the difference between these two expressions is every term between $(n-k+1)$ and (1), which are equal to $(n-k)!$, we can simply divide $n!$ by $(n-k)!$ to give the final multiplier we are looking for:

$$\frac{n!}{(n-k)!}$$

Now, all that is left is to amalgamate each of these parts, creating the complete formula for the number of k-cubes present within an n-cube:

$$\left(\frac{n!}{(n-k)!}\right)\left(\frac{2^n}{2^k k!}\right)$$

5. In simple terms...

The simplification of this formula provides a surprisingly and satisfyingly elegant solution, which we can reach as follows:

Moving the $k!$ from the second denominator to the first gives:

$$\left(\frac{n!}{(n-k)!\,k!}\right)\left(\frac{2^n}{2^k}\right)$$

And now we can simplify the second bracket to:

$$\left(\frac{n!}{k!\,(n-k)!}\right)(2^{n-k})$$

Seem familiar? The formula we have found resembles the binomial theorem – a method of easily expanding powers of a binomial. According to the theorem, the expansion of the binomial $(x + y)$, to a power n, will give the following[3]:

$$(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n-1}x^1 y^{n-1} + \binom{n}{n}x^0 y^n$$

Each $\binom{n}{k}$ is called the binomial coefficient, which is defined in the theorem as:

$$\left(\frac{n!}{k!\,(n-k)!}\right)$$

...which matches the first bracket of the equation we found. As for the rest of the theorem, each x term can be expressed as y^{n-k}, similar to the 2^{n-k} in our formula. Therefore, we can substitute the y in $(x+y)^n$ to a 2, leaving $(x+2)^n$.

This means that the expansion of this formula, $(x+2)^n$, where n is the number of dimensions in the hypercube, will give a *complete* catalogue of the elements of that hypercube, in the form that each coefficient of x^k is the number of elements of dimension k. We can test this once more with some examples of which we already know the properties:

If we look at $n = 2$ (a 2-dimensional square) we find that:

$$(x+2)^2 = x^2 + 4x + 4$$

Telling us that a square is comprised of four 0-dimensional elements (vertices), four 1-dimensional elements (sides), and one 2-dimensional element (the square itself).

We can do the same for a cube ($n = 3$):

$$(x+2)^3 = x^3 + 6x^2 + 12x + 8$$

Telling us that a cube is comprised of 8 vertices, 12 sides, and 6 faces ($k = 2$).

Of course, this can be used to predict the characteristics of more complex hypercubes, allowing us to model their theoretical shapes.

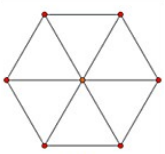
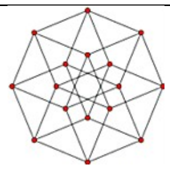
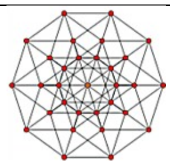
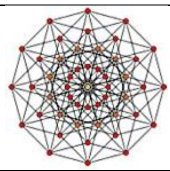
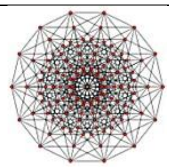
Name	*3-Cube*	*4-Cube*	*5-Cube*	*6-cube*	*7-cube*
Model					

Figure 7: Orthogonal projections of hypercubes[4]

[3] Source: https://en.wikipedia.org/wiki/Binomial_theorem

[4] Images source: https://character-level.fandom.com/wiki/Dimensional_Tiering

6. The triangle of cubes

Another interesting facet of this formula for the elements of hypercubes arises from the link between the binomial coefficient, and Pascal's triangle (seemingly mystical in its myriad of unexpected applications). As well as each term being the sum of those above it, Pascal's triangle is a representation of the coefficients in the expansion of a binomial – meaning it can tell us the properties of hypercubes.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Figure 8: Pascal's Triangle

To find the binomial coefficient $\binom{n}{k}$, you can simply use the k^{th} entry of the n^{th} row of the triangle. Since our formula includes 2^n, and the number of an element involves the coefficient of the respective x term, all that is needed to go from the triangle to a hypercube is to multiply the n^{th} row of the triangle by 2^{n-k} (the top row is the 0^{th} row). This will, once again, give the numbers of elements contained within that n-cube. For example, taking the fourth row, $1\ 4\ 6\ 4\ 1$, and multiplying each term by 2^{n-k}, gives:

$$1 \times 2^4, 4 \times 2^3, 6 \times 2^2, 4 \times 2^1, 1 \times 2^0 \ = \ 16, 32, 24, 8, 1$$

Representing the tesseract's 16 vertices, 32 edges, 24 faces, 8 cells (a 3-dimensional element of a hypercube, equivalent to a normal cube), and 1 tesseract itself.

7. Conclusion

In this essay I have explored the elegant way an expression as simple as $(x+2)^n$ can hold such complicated information about the composition of multidimensional shapes. I have attempted to explain using just logic and our pre-existing comprehension limited to our own three dimensions, by extrapolating the patterns we can observe. Although I have barely scratched the surface of this expansive topic, hopefully this method of explanation can introduce an intuitive element to a topic that is otherwise incompatible with intuition. I found that, despite the intangible nature of hypercubes, and their elusive, inconceivable forms, there is still something beautiful in their conception through the lens of such an unassuming formula.

Additional References:

- https://www.physicsforums.com/threads/creating-a-4-dimensional-cube.903565/
- https://math.stackexchange.com/questions/442349/surface-area-of-a-hypercube
- https://mathworld.wolfram.com/Four-DimensionalGeometry.html
- https://en.wikipedia.org/wiki/Binomial_theorem