The Birthday Paradox

How many people need to be in a room such that there is a greater than 50% chance that 2 people share the same birthday? In general, people's initial thought is; there are 365 days in a year, at least half that many people need to be in the room, so you need 183 people. You might find it hard to believe but you only need 23 people in the room. Sometimes what seems logical turns out to be proved false with math. I have chosen this topic as I didn't believe it at first. I was very skeptical and thought that it didn't make sense and therefore the problem was incorrect. I then researched it and found out I was wrong, I think that this is a very interesting topic that can surprise people.

To begin, I have made the assumption that every year has 365 days. As, the inclusion of leap years would lower the given probabilities slightly. I have also made the assumption that there is an equal chance for any birth date. In addition, birthday means the month and day someone was born, regardless of the year.

Birthday Problem

This question can be simplified by asking the opposite question. What is the probability that, in a room with 23 people, no one shares a birthday.

1. One person enters the room

As there is only one person the probability that no one shares their birthday is p1=$\frac{365}{365} = 1$

2. A second person enters the room

Person 1 has a birthday on one day in the year. Person 2 must have their birthday in any of the other 364 days of the year in order to not share with person 1. The probability that person 2 has a different birthday is p2 $=1 \times \frac{364}{365}$. Meaning that the probability that they have the same birthday is

$p = 1 - \frac{364}{365} = \frac{1}{365} \approx 0.274\%$

3. A third person enters the room

Again, person 1 has a birthday at one day, person 2 has a birthday on a different day

p2 $= \frac{364}{365}$. For person 3 to have their birthday on a different day than person 1 and person 2, person 3's birthday must be one in the 363 days left with a probability of $\frac{363}{365}$. The probability that all three people have varying birthdays is

$p'3 = (1 \times \frac{364}{365} \times \frac{363}{365}) \times 100 \approx 99.17\%$.

Which implies that the probability that at least two of them share the same birthday is

$p = 1 - p'$

$p = (1 - (1 \times \frac{364}{365} \times \frac{363}{365})) \times 100 \approx 0.82\%$

4. When the twenty-third person enters the room

$p23 = \frac{365-23+1}{365} = \frac{343}{365}$

The probability that none of the 23 people share a birthday is calculated by the following formula:

$p' = (1 \times \frac{364}{365} \times \frac{363}{365} \times [...] \times \frac{343}{365})$

$p_n = \frac{365 \times 364 \times 363 \times [...] \times 365 - n + 1}{365^n}$

$= \frac{365!}{365^n \times (365-n)!}$

Explanation

The first line and second line in the formula equal each other ($p' = p_n$). The reason that the end of formula p_n is 365-n+1 can be seen in previous examples. For example, when the third person enters the room there are 363 days left (363-3+1) for them to not share a birthday. Then "365^n" is used because as seen in p' the denominator stays as 365 and is multiplied the same amount of times as the number of people in the room. In the third line there is an exclamation mark (!). This is called factorial, factorial is the operation of multiplying any full number with all the full numbers that are smaller than it. In this case $365! = 365 \times 364 \times 363 \times [...] \times 2 \times 1$. The multiplication on the numerator stops at 365-n+1, for example when there are three people in the room ($\frac{365\times364\times363}{365\times365\times365}$) the number on the denominator stops at $363 = 365 - 3 + 1$. To cancel out the numbers lower than 365-n+1 I placed (365-n)! as the denominator.

Solution

The birthday problem states that there must be 23 people in a room in order for there to be a greater than 50% chance that 2 people share the same birthday. We know that the probability that 2 people *don't* share a birthday in a room with n amount of people is calculated with this formula:

$$p_n = \frac{365!}{365^n \times (365-n)!}$$

In order to calculate the probability that 2 people *don't* share a birthday in a room with 23 people we substitute n with 23.

$$p_{23} = \frac{365!}{365^{23} \times (365-23)!}$$

Using a calculator to calculate this, it gives an answer of

≈ 0.493

$0.493 \times 100 = 49.3\%$

To calculate the probability that at least 2 people share a birthday in a room with 23 people

$P = 1 - p_n$

$P = 1 - 0.493$

$= 0.507$

$0.507 \times 100 = 50.7\%$

Which means that the probability that someone does share a birthday is 50.7%

This can be confusing to grasp as humans are "self-centered". When asked this question most people think about themselves walking into a room and finding someone with their own birthday. In a room with 23 people, that means that there are 22 people that can match with you. However, the problem means that any pair of people in the room have the same birthday. In a room with 23 people there is a total of 253 pairs possible

$\frac{23\times22}{2} = 253$

Solving the 2nd birthday paradox

I am going to solve the "self-centered" point of view to see the difference between the two paradoxes. What is the probability that in a room filled with 23 people at least one of them has birthday x?

1. One person enters the room

Person 1 has a birthday one day of the year but it cannot be birthday x. The probability that it is not the same day is $p1 = \frac{364}{365}$. The probability that it is birthday x is

$p = 1 - \frac{364}{365} = \frac{1}{365} \approx 0.274\%$

2. A second person enters the room

Person 1's birthday is not birthday x and the probability that person 2's birthday is not birthday x is $p2 = \frac{364}{365}$. This means that the probability that neither person 1 nor person 2 have birthday x is

$$p' = (\frac{364}{365})^2 \times 100 \approx 99.5\%$$

The probability that at least one of them have birthday x is

$$p = 1 - p'$$

$$p = (1 - (\frac{364}{365})^2) \times 100 \approx 0.5\%$$

This formula can be used in terms of n

$$p_n = 1 - (\frac{364}{365})^n$$

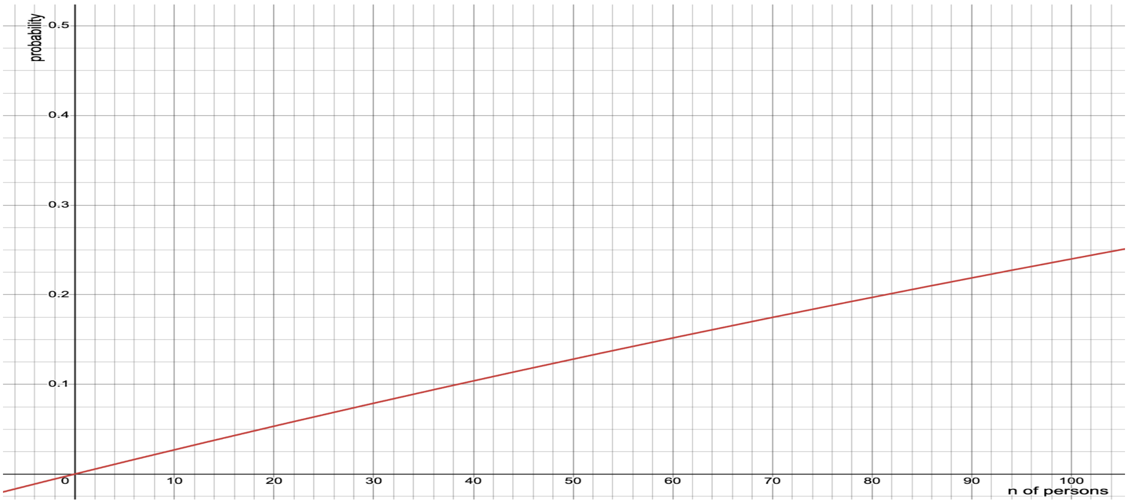

Figure 1: Probability that in a room of n persons at least one has birthday x.

Figure 1 helps visualize the little chance that someone in a room with n people have birthday x. Taking that into account, what is the probability that in a room with 23 people at least one has birthday x?

$$p_n = 1 - (\frac{364}{365})^n$$

$p_{23} = (1 - (\frac{364}{365})^{23}) \times 100 \approx 6.12\%$

The probability that someone in a room with 23 people has the same birthday as you (birthday x) is 6.12%.

To answer the other question, how many people need to be in a room such that there is a greater than 50% chance that at least one person has birthday x?

$p_n = 1 - (\frac{364}{365})^n$

$p_n = 50\%$

$(1 - (\frac{364}{365})^n) \times 100 = 50\%$

$= 252.652$

$= 253$

There must be 253 people in a room in order for there to be a greater than 50% chance that at least one person has birthday x. This number is curious as 253 is also the number of pairs possible in a room with 23 people. This could be an interesting topic to investigate further.

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