Needles, Noodles, and π: Buffon's Elegant Estimation

Picture this: you've got a needle and lined paper, and you decide to drop the needle onto the lined sheet of paper. What may seem like a trivial act of dropping the needle is, in fact, a surprisingly profound experiment used to estimate the most famous mathematical constant, π.

This is the essence of Buffon's Needle Problem; conceived by French Mathematician Leclerc, Comte de Buffon. This classical problem presents an ingenious method to estimate π through a simple yet effective probability exercise. Suppose we have a lined sheet of cardboard, each with the same spacing, measuring the length of 2 needles (16 cm), and we drop 100 needles randomly onto the paper. What is the probability that the needle will lie across a line?

Part 1: The Estimation Game

Figure 1 My estimation.

If we now divide the total number of needles used (100) by the number of needles completely crossing a line (32), we should verge on a number close to π. $\frac{100}{32} = 3.125$, resulting in a 0.53% deviation from π, close enough to see a correlation, the only setback being a relatively small sample size. To demonstrate the impact, I compiled a simulation on Wolfram in Mathematica with the same conditions as my estimation, except I tried with 1000 needles.

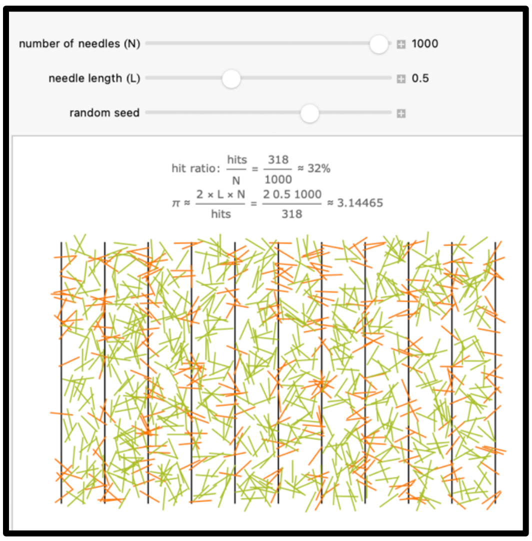

Figure 2 Wolfram's Estimation

The Wolfram simulation achieved an estimation correct to 2 decimal places (3.14). In contrast, a mathematician called Lazzarini used 3,408 needles and achieved an estimation of π to 6 decimal places (3.141592), without technology. But why does this experiment verge on π?

Buffon's needle was one of the earliest geometric problems to be solved. Two notable approaches have emerged for solving it. The first approach applies a probability density function and involves complex double integrals. The second approach, which we will explore, is a slight variation on the original, known as Buffon's noodle which builds on from the proof of Buffon's needle, introduced by Joseph-Émile Barbier in 1860, which also serves to prove Barbier's theorem.

Part 2: The Needle Proof

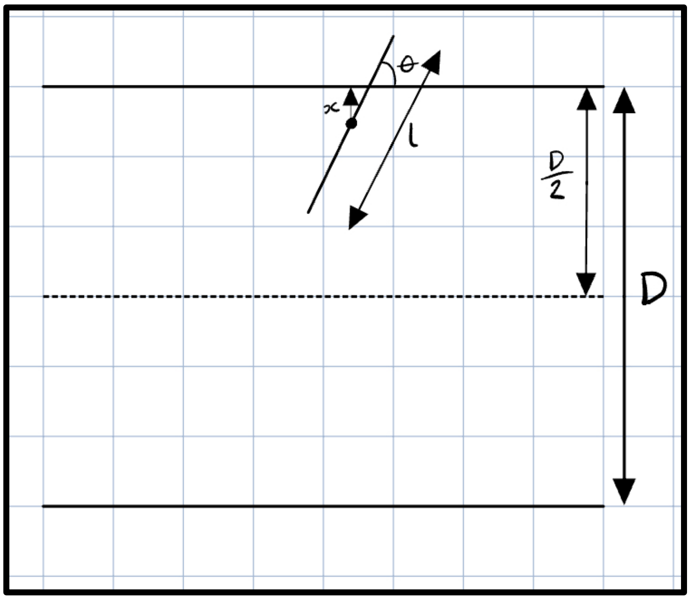

Figure 3 Buffon's needle scenario.

For Buffon's needle, imagine a slightly different scenario from our own estimation where the lines can be spaced any length, and the needles can be of any length too. The needle is of length l, the distance between the lines is D, the distance from the centre of the needle to the closest line is x, and the angle formed between the needle and parallel lines is θ. What are the conditions for the needle to cross a line?

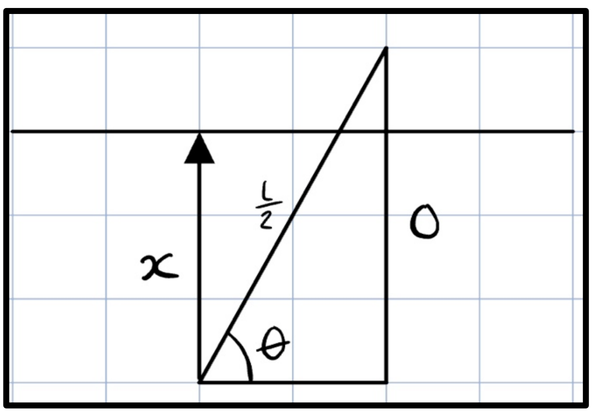

Figure 4 Conditions for the needle to cross the line.

The needle crosses the closest line if x is less than $\frac{l}{2} * \sin(\theta)$. If x is the distance from the centre of the needle to the closest line, the length of the needle must be $\frac{l}{2}$. From this we can form a right-angled triangle to make use of the sine function. $\text{Sin}(\theta) = \frac{opposite}{hypotenuse}$. In this case, the hypotenuse would be $\frac{l}{2}$ and we can mark "opposite" as O. Thereby we can formulate $\frac{l}{2} * \sin(\theta) = O$. For the needle to cross the line, O must be greater than or equal to x. This can be proven and visualised in Figure 3, where the needle crosses the line because O is greater than or equal to x.

The values for x and θ are randomly determined each time the needle is dropped. However, there are possible ranges for the values of x and θ. The possible values for x are $0 \leq x \leq \frac{D}{2}$, the minimum value of x occurs when the needle's centre aligns with a line, and the maximum value occurs when it lies precisely midway between two lines. The possible values for θ are $0 \leq \theta \leq \frac{\pi}{2}$ (because we only consider the acute angle). Therefore, the sample space, which is the set of all possible outcomes, for x, θ is thus a rectangle of side lengths $\frac{D}{2}$ *and* $\frac{\pi}{2}$ in the $'x$ - θ' plane.

To find the probability of the needle crossing a line, we need to calculate the ratio of the successful area (when x < O) over the total sample space. However, we must complete one basic integration, to sum up all the possibilities for different angles and positions for when the needles cross a line. Integration is used because we are dealing with continuous probability space.

To find the total successful area we must integrate $\frac{l}{2} * \sin(\theta)$ with respect to θ (our limits are 0 and $\frac{\pi}{2}$).

$$\int_0^{\frac{\pi}{2}} \frac{l}{2} \sin(\theta)\, d\theta$$

$$\frac{l}{2} \int_0^{\frac{\pi}{2}} \sin(\theta)\, d\theta$$

$$-\frac{l}{2} \cos\left(\frac{\pi}{2}\right) + \frac{l}{2} \cos(0) = \frac{l}{2}$$

We can take the constant $\frac{l}{2}$ out the integration and integrate $\sin(\theta)$ to $-\cos(\theta)$. With our sample space area simplifying to $\frac{D\pi}{4}$, we can divide the successful area by the sample space area to equal the probability of a needle landing on a line.

$$P = \frac{Area\ (successful)}{Area\ (sample\ space)} = \frac{l}{2} * \frac{4}{D\pi} = \frac{2l}{D\pi}$$

If we now plug-in the values from my own estimation $\frac{2(8)}{16\pi}$, this simplifies down to $\frac{1}{\pi}$. Therefore, we can multiply 100 by $\frac{1}{\pi}$ to get approximately the number of matches that should cross the line, which computes to $31.83 \approx 32$ (the number which we counted!).

Part 3: Why the name "noodle"?

Does the formula stay the same even when the noodle is bent? If you imagine breaking the needle into 2 straight segments, known as piecewise linear, we can consider each segment's crossings independently. The expectation of the total number of crossings is the sum of the expected crossing for each segment due to the linearity of expectation (the expected number of times the entire noodle crosses the lines is simply the sum of the expected number of times each individual segment crosses the line). To illustrate, suppose you have two connected segments, A and B. The expected number of crossings for A and B is 1. Even if B's position depends on where A ends, the total expected number of crossings for both segments combined is 2. This holds true no matter how the segments are connected, only if you can define the expected number of crossings for each segment individually.

What happens if we want to find the expected number of crossings for a completely curved noodle, where it's difficult to break it into small segments. How does the average number of crossings depend on shape? To work this out consider the circle below of diameter 1.

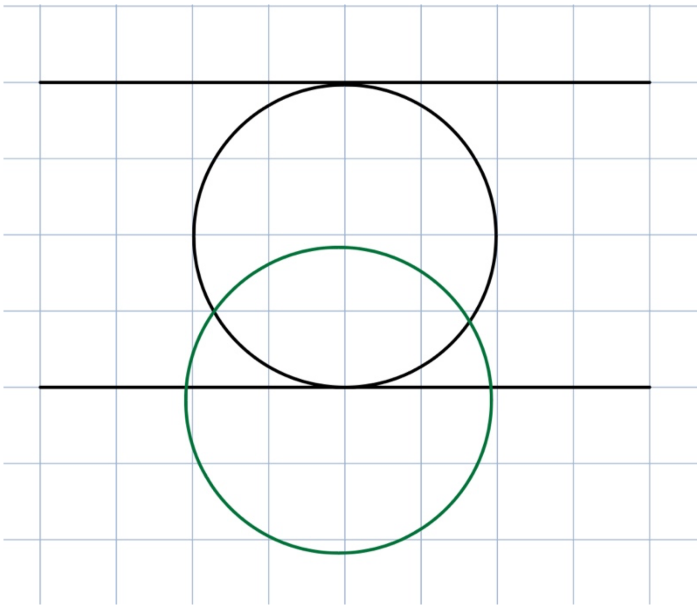

Figure 5 Various Positions of Circles on lines.

Wherever the circle moves, it intersects the equally spaced parallel lines twice (two examples shown in green and black). Regarding a curved noodle as the limit of a sequence of piecewise linear noodles (infinite piecewise linear noodles), we conclude that the expected number of crossings is proportional to the length.

If we then plug-in values from the circle with a circumference of π (same as length) and a diameter of 1 (distance between parallel lines)

$$\frac{2\pi}{\pi} = 2$$

We see that this formula holds for completely curved noodles as well, providing us with an expected probability of number of crossings per noodle. Of course, this is only an expected value.

Part 4: Proof of Barbier's theorem

Barbier's theorem states every curve with constant width 1 must have perimeter π. The classical proof is notoriously challenging, but within our noodle setting we can think of them as noodles thrown onto equalled spaced lines.

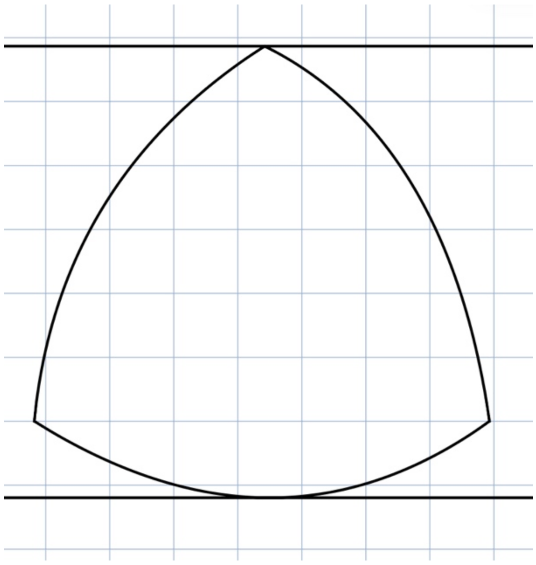

Figure 6 Reuleux Triangle (shape with constant width) on lines.

With constant width being equal to D, the distance between the parallel lines, as the curve crosses the line twice. If we plug-in the values of 1 for D and 2 for the probability of expecting crossings

$$\frac{2l}{\pi} = 2$$

$$l = \pi$$

We can re-arrange to proof the perimeter of the curve is π!

Part 5: The Conclusion

As we conclude this exploration, I hope that this essay has done more than just acquaint you with the peculiarities of Buffon's Needle and its unexpected connection to π. It is my hope that these words have broadened your perspective on mathematics, demonstrating that even a mundane act, the toss of cocktail sticks onto lined paper, can yield estimations of a number as enigmatic as π and validate theorems of significant consequence.

The simplicity of Buffon's experiment is deceptive. It serves as a metaphor for the broader mathematical landscape, where beneath the surface of basic operations lies a complex web of principles. Just as the sticks fall in random yet predictable patterns, so too does mathematics reveal its consistencies in the chaos of mathematics.