“By one more than the previous one” – a primer on Vedic mathematics

Introduction - a striking example

I get it – skip the maths working, and those teachers might just hunt you down. After all, those scribbles on paper hold the key to scoring marks. However, allow me to present the following deceptively simple exam question:

Convert the fraction $\frac{1}{19}$ into a recurring decimal.

At first glance, it might not seem too daunting. Tradition guides us to employ the age-old “bus-stop” method, the sole technique bestowed upon us and handed down through generations of mathematical tradition.

The journey begins with seemingly simple calculations: when 19 fails to divide into 1, we turn 10. Yet even 10 doesn’t work, leading us to 100. Dividing 100 by 19 gives us five sets of 19, totalling 95. Subtracting 95 from 100 leaves us with 5, marking the start of the next digit in the sequence.

Soon, you’re hit with a chilling realisation – a daunting task of 18 intricate steps sprawls before you. The reason lies in the 18 recurring digits of the decimal answer, requiring 18 divisors to conquer, 18 subtractions to navigate, and 18 instances of beckoning down the next digit in the sequence.

Navigating through a million potential pitfalls, a single misstep could cascade into a chain reaction of errors, and then the whole problem becomes an utter waste of time.

Now imagine solving this problem in just one line, without showing any working. You might as well as you close your eyes and pray the question was a typo. Enter the Vedic maths sutra, *Ekadhikena Purvena*, which translates to “by one more than the previous one”.

But this is maths, not English class. What does this metaphorical language have to do with this? Well, the preposition “by” implies either multiplication or division. It actually turns out either can be applied. Let’s delve into its practical application.

We put the 1 as the rightmost digit in our answer (the last digit in the denominator is the 9, and so the 1 before it is interpreted as the “previous one”).

1

We multiply that digit by “one more”, i.e. 2, and put the result (2) down as the immediately preceding digit.

2 1

We multiply that 2 by 2 and put 4 down as the next preceding digit.

$$4\,2\,1$$

We continue in this fashion until we reach 18 digits. Where the multiplication produces a two-digit number, we carry the first digit, like so:

$${}_1 6\,8\,4\,2\,1$$

We then multiply the 6 by 2, get 12, and then add 1, to get 13 as the consolidated product:

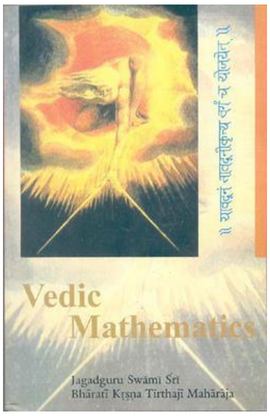
$${}_1 3\,{}_1 6\,8\,4\,2\,1$$

We follow this procedure until we reach the 18th digit, counting leftwards from the right, when we find that the whole decimal has begun to repeat itself:

$$\frac{1}{19} = 0\,.{}_1 0\,5\,{}_1 2\,6\,3\,{}_1 1\,{}_1 5\,{}_1 7\,{}_1 8\,9\,{}_1 4\,7\,{}_1 3\,{}_1 6\,8\,4\,2\,1$$

So, we find that the answer we get through our Vedic one-line mental arithmetic is identical to what you would get using the bus-stop method. But, if you're still plodding along with that, you wouldn't know that, would you?

Interestingly, there is another shortcut that cuts the workload in half, which I'll cover in the section exploring the versatility of this sutra.



A brief history of Vedic mathematics

Discussions on Vedic mathematics often refer to a book of the same name penned by Indian scholar Shankaracharya Bharati Krishna Tirtha. In this book, he presents metaphorical aphorisms called sutras, along with thirteen sub-sutras, forming the foundation of his mathematical system. Krishna Tirtha claims these sutras were revealed to him after years of studying the Vedas, sacred Hindu scriptures, hidden within the *parisista*, a supplementary section of the Vedas. However, sceptics question the existence of these sutras in Vedic literature, noting the book's linguistic style resembles modern Sanskrit more than ancient Vedic texts. Despite this, the term "Veda" has a derivational meaning, a "limitless storehouse of knowledge", suggesting that the Vedas *should* encompass all knowledge essential for humanity, including both spiritual and "worldly" matters (which includes mathematics).

While some attribute Krishna Tirtha's methods to his mathematical training and experimentation with numbers, similar systems like the Trachtenburg system and European treatises on calculation also exist. Regardless of their origins, these methods seek to present an integrated approach to learning maths, with a focus on pattern recognition and inquisitiveness. Krishna Tirtha aims for school children to avoid the monotony of accepting theories and working from them mechanically.

The sutras: building blocks of Vedic mathematics

Vedic mathematics is characterised by its metaphorical aphorisms, encapsulated in sixteen sutras and thirteen sub-sutras. An aphorism is simply a short, concise statement that expresses a wise or profound idea. These literary expressions, which can be compared to abstract poetry, invite creative interpretations, allowing for a multitude of mathematical equivalencies across diverse contexts.

Below are some attributes of the sutras outlined in Krishna Tirtha's preface:

- The sutras encompass and address every aspect of all mathematical disciplines.
- They are straightforward to understand, apply, and commit to memory.
- Utilising the Vedic approach typically consumes a fraction of the time compared to "Western" methodologies.
- Though solutions appear like magic, the application of the sutras is perfectly logical and rational. Krishna Tirtha aptly notes: "It is magic until you understand it; and it is mathematics thereafter."

Exploring the adaptability of sutras across diverse problem types

To illustrate the versatility of the sutras, let us revisit the *Ekadhikena Purvena* sutra (which, as a reminder, translates to "by one more than the previous one"), and examine its application in two distinct contexts. I provide algebraic justification in both scenarios to prove why they work.

1) Converting fractions whose denominators end in a 9 into recurring decimals.

We have already come across the example of 1/19, but here are a few interesting observations we could make that we didn't point out earlier:

- The division process itself is not used at all.
- For any fraction where the units digit of the denominator is a 9, the rightmost digit in the recurring decimal is always a 1.
- The sum of the first digit and the last digit in the recurring digits block is equal to 9 (1 + 8 = 9). The sum of the second digit and the penultimate digit is also equal to 9 (2 + 7 = 9). We may conclude that if we find the first nine digits, the rest of them can be derived as complements of 9. Thus, we can reduce the work needed by a further 50%.

Algebraic proof:

A fraction whose denominator ends in the digit 9 can be expressed as: $\frac{1}{(a+1)x-1}$, where a represents any integer and $x = 10$ (substitute in a couple of values for a and see what fractions you get if you're a little unsure).

We can factorise out $(a+1)x$ from the denominator, although it might be a little tricky to see how:

$$\frac{1}{(a+1)x-1} = \frac{1}{(a+1)x\left[1-\frac{1}{(a+1)x}\right]}$$

$$= \tfrac{1}{(a+1)x}\left[1-\tfrac{1}{(a+1)x}\right]^{-1}$$

Using a power series expansion, we can write this as:

$$= \tfrac{1}{(a+1)x}\left[1+\tfrac{1}{(a+1)x}+\tfrac{1}{(a+1)x^2}+\tfrac{1}{(a+1)x^3}+\cdots\right]$$

$$= \frac{1}{(a+1)x}+\frac{1}{(a+1)^2x^2}+\frac{1}{(a+1)^3x^3}+\cdots$$

Substituting in $x = 10$, this becomes:

$$= 10^{-1}\left(\frac{1}{(a+1)}\right)+10^{-2}\left(\frac{1}{(a+1)^2}\right)+10^{-3}\left(\frac{1}{(a+1)^3}\right)+\cdots$$

Now consider the problem of 1 / 19. From above, we get after substituting in $a = 1$:

$$\frac{1}{19} = 10^{-1}\left(\frac{1}{(1+1)}\right)+10^{-2}\left(\frac{1}{(1+1)^2}\right)+10^{-3}\left(\frac{1}{(1+1)^3}\right)+\cdots$$

$$= 10^{-1}\left(\tfrac{1}{2}\right)+10^{-2}\left(\tfrac{1}{2}\right)^2+10^{-3}\left(\tfrac{1}{2}\right)^3+\cdots$$

$$= 10^{-1}(0.5)+10^{-2}(0.25)+10^{-3}(0.125)+\cdots$$

$$= 0.05+0.0025+0.000125+0.00000625+\cdots$$

$$= 0.052631\ldots$$

Proving the sutras isn't a major focus of this essay. What really matters is recognising that the methods described *can* be proven (not witchcraft!), and that they rely on pattern recognition across different mathematical fields, rather than specific formulas. In fact, this technique can be applied to some fractions whose denominators do not end in a 9, e.g.:

$\frac{1}{7} = \left(\frac{7}{7}\right)\cdot\left(\frac{1}{7}\right) = 7\cdot\frac{1}{49}$, where $\frac{1}{49}$ can be found out using the method above.

Similarly, $\frac{1}{13} = \left(\frac{3}{3}\right)\cdot\left(\frac{1}{13}\right) = 3\cdot\left(\frac{1}{39}\right)$

2) Finding the squares of numbers ending in 5.

A completely random problem, seemingly unrelated to the first context. Let us take the example of 25. For this number, the last digit is a 5, and the "previous" digit is a 2. Hence, “one more than the previous one” can be evaluated as 2 + 1 = 3. The Sutra, in this context, gives the procedure “to multiply the previous digit by one more than itself, that is, by 3”. This becomes the “left hand side” of the result if you like, that is 2 x 3 = 6. The “right hand side” is 5 squared, that is, 25.

Thus, $25^2 = [2 \cdot (2+1)], [25] = 625$.

In the same way,

$$35^2 = [3 \cdot (3+1)], [25] = 1225$$

$$65^2 = [6 \cdot (6+1)], [25] = 4225$$

Algebraic proof:

Consider the identity $(ax+b)^2 \equiv a^2x^2 + 2abx + b^2$.

For two-digit numbers, we can substitute in $x = 10$ and $b = 5$, and so the identity becomes:

$$\begin{aligned}(10a+5)^2 &= a^2(10^2) + 2a(10)(5) + 5^2 \\ &= 100a^2 + 100a + 25 \\ &= 100(a^2 + a) + 25 \\ &= 100a(a+1) + 25\end{aligned}$$

$10a + 5$ represents two-digit numbers that end in 5, i.e. 15, 25, 35, all the way to 95, for the values a = 1, 2, 3, ... , 9. The number $(10a+5)^2$ is of the form whose final two digits make 25 and the digits to the left of it are of the form $a(a+1)$, i.e. by one more than the previous one. We can show a similar result for three-digit numbers which end in 5 using the identity $(ax^2 + bx + c)^2 = a^2x^4 + b^2x^2 + c^2 + 2abx^3 + 2bcx + 2acx^2$, but that’s a little more tedious.

Embracing a new mindset

The Vedic maths system stands out for its flexibility, offering various methods for calculations rather than a single, rigid approach. This flexibility allows students to choose and develop their own methods, developing creativity and intuition.

Modern teaching methods can be rigid and boring. The Vedic system provides a more engaging approach that intelligent and creative students are less likely to rebel against.

Moreover, the system is coherent and interconnected, and not simply a collection of unrelated techniques. Once students master an arithmetic method, they can easily apply it to algebraic problems of a similar nature. This coherence between arithmetic and algebra enhances the understanding and application of mathematical concepts within the Vedic system.

The application of the *Vilokanam* sutra, meaning "observation", vividly demonstrates this concept. Take, for instance, the equation:

$$x + \frac{1}{x} = \frac{5}{2}$$

The typical approach to solving this equation involves multiplying both sides by x and proceeding as follows:

$$x + \frac{1}{x} = \frac{5}{2}$$

$$x^2 + 1 = \frac{5}{2}x$$

$$x^2 - \frac{5}{2}x + 1 = 0$$

$$2x^2 - 5x + 1 = 0$$

$$(x - 2)(2x - 1) = 0$$

$$x = 2 \text{ or } x = \frac{1}{2}$$

However, applying *Vilokanam*, $x + \frac{1}{x} = \frac{5}{2}$ can be viewed as:

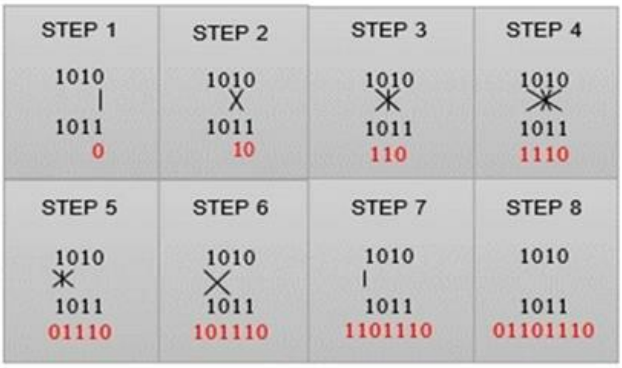
$$x + \frac{1}{x} = 2 + \frac{1}{2}$$

Through simple observation, we identify the two solutions, $x = 2$ or $x = \frac{1}{2}$.

This principle extends to more complex equations, but the essence remains the same.

Conclusion

In this essay, we embarked on a tour of Vedic mathematics, delving into its historical roots and the concept of sutras. We explored the *Ekadhikena Purvena* sutra in detail, including its application in two diverse contexts along with supplementary proofs. However, this journey merely scratched the surface of the vast landscape of Vedic mathematics. With fifteen other sutras, thirteen sub-sutras, and realms such as calculus and simultaneous equations left unexplored, the potential for further discovery and application remains. Indeed, abstract Vedic sutras are finding their way into real-life applications, with developments such as the Vedic multiplier circuit modelled upon the *Urdhva Tiryagbhyam* sutra for efficient multiplication. While Vedic maths is unlikely to supplant the current maths education system, it's always fascinating to explore alternative perspectives on mathematical concepts.



The application of the Urdhva Tiryagbhyam sutra (meaning "vertically and crosswise") to multiply two 4-bit numbers. A bit represents the smallest unit of storage in a computer, a 1 or a 0. Computers use the binary system to represent information, where each place value doubles from right to left. For example, in a 4-bit system, the values are 1, 2, 4 and 8. So, the maximum number representable using four bits is 15 (1 + 2 + 4 + 8).

References

Wikipedia (2024) *Vedic Mathematics*. Available at: https://en.wikipedia.org/wiki/Vedic_Mathematics#:~:text=Vedic%20Mathematics%20is%20a%20book (Accessed: 26 March 2024).

Sri Sathya Sai Veda Pratishtan (n.d.) *Vedic Mathematics – Methods.* Available at: http://www.vedamu.org/veda/1795$vedic_mathematics_methods.pdf (Accessed: 26 March 2024).

Chugh, H. (2022) *Design and Implementation of a High-Performance 4-bit Vedic Multiplier Using a Novel 5-bit Adder in 90nm Technology.* Available at: https://www.researchgate.net/publication/366139161_Design_and_Implementation_of_a_High-Performance_4-bit_Vedic_Multiplier_Using_a_Novel_5-bit_Adder_in_90nm_Technology (Accessed: 26 March 2024).