Examining the Historical Context and Mathematical Significance of Fermat's Last Theorem (FLT) : How Mathematical Collaboration Drives Meaningful Proofs

What is FLT?

To begin with, what really is Fermat's Last Theorem? The formal theorem states that, for a positive integer n, greater than 2, and any 3 natural numbers, a, b, c, the following equation will never be satisfied for all combinations of a,b,c,n in their respective sets:

$$a^n + b^n = c^n$$

Figure 1 : Fermat's Last Theorem

Perhaps an easier way to visualise this otherwise seemingly abstract equation, is that it is impossible for the volume of a cube with natural number side lengths to be the same as the volume of two smaller cubes with smaller natural number side lengths. As n can be any integer > 2, the theorem suggests the same for the equivalent of a cube in 4th dimensional space, 5th, 6th etc.

Why does this seemingly abstract hypothesis hold such substance, and how does it highlight the necessity of collaboration within Maths? On top of its historical significance, which I will examine more closely later, the proof of Fermat's Last Theorem has had resounding significance in mathematical fields like algebraic geometry and number theory. However, this significance was not predominantly rooted in the new knowledge that Fermat's Last Theorem was indeed correct. On the contrary, the magnitude and gravity of the new mathematics that was created as a result of the journey towards the proof is what held much deeper and wider implications.

Mathematical Significance

The methods that Andrew Wiles and other mathematicians utilised in proving the conjecture have provided innovative skills to the toolsets of many research mathematicians. From this, we can derive real-world benefits such as the generalised link between modular forms and elliptic curves, the full modularity theorem, which we will explore shortly. Further, the historical context and long-standing nature surrounding the theorem had rendered it seemingly impossibly complex to prove. As such, its eventual solution demonstrated that even the most complex problems can be solved with persistence and creativity, giving rise to new efforts into consequential unsolved problems.

Examining the Modularity Theorem and the Langlands Program

A prime example of this, in the context of number theory specifically, is the Modularity Theorem and Langlands Program in a broader sense. The Langlands program is an extremely large, ambitious and consequential project in mathematics. It can be seen as a unification of the analytic and algebraic sides of number theory. Very simplified, it gives rise to non-trivial symmetries of objects defined by particular infinite sequences. These symmetries can be referred to as a case of automorphism. An automorphic function will revert back to itself upon changing its variable by some process. Taking the well-known sine function as an example, when we slide the argument of the function by 2 pi, it reverts back to itself.

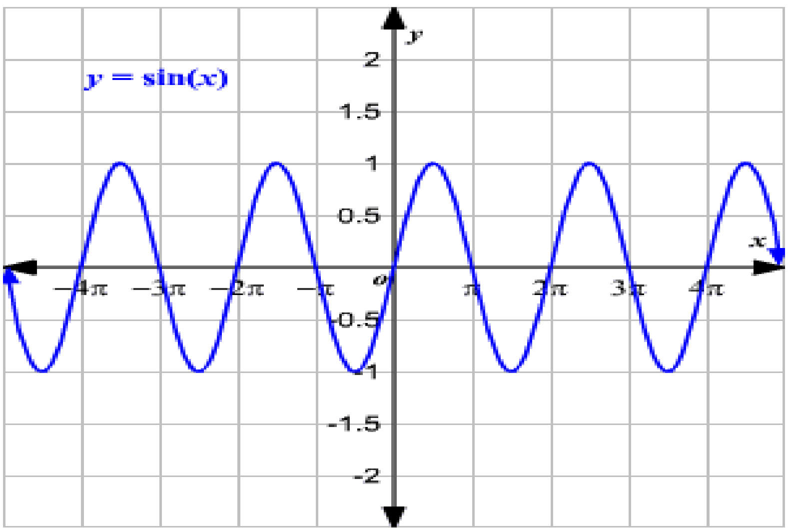

Figure 2 : Our beloved sine function

But, what is the significance of these symmetries in the context of mathematical advancements? To get an idea for how the relationships provided by the Langlands Program can help solve sets of complex, obscure, yet potentially momentous problems, we can look at the Ramanujan conjectures. Generalised, the Ramanujan conjectures are still unsolved to this day. If solved, they could provide profound advancements into the theory of modular forms and their connections to number theory. This could have knock-on effects into other subject areas such as Physics or Computational mathematics, with modular forms holding deep ties with string theory, field theory and computational algorithms. The Ramanjuan conjectures roughly state that an automorphic function given by some sequence of coefficients like so:

$$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

Figure 3 : Ramanujan Conjectures

must have all coefficients (a, a1, a2 …) to be bounded between -1 and 1 inclusive.
Again, we have been unable to prove this so far. The most restrictive limit for the value of the coefficients that has been proven, is that they are bounded by 10. This seems almost arbitrary and pointless, in regards to the original bounds that are conjectured to be true. However, certain potential

advancements in the Langlands Program can take this seemingly arbitrary and pointless result, and transform it into a relatively simple proof. Crucially, the Ramanjuan conjectures are infinite sequences. Going back to the original definition that we have made for the Langlands program, if we are able to prove that functoriality is one of the symmetries of the Langlands Program, then we can prove the Ramanujan conjectures using this bound of 10. Functoriality claims that G(x) in Figure 2 can give rise to other automorphic functions (G2(x)) by simply raising all of the coefficients to the same integer power.

$$G_2(x) = {a_0}^2 + {a_1}^2 x + {a_2}^2 x^2 + {a_3}^2 x^3 + \ldots$$

Figure 4 : Ramanujan Conjectures with functoriality shift

G2(x), since it is automorphic with G(x), will have the coefficients of each term also bounded by 10. Now, the crux of the proof! We can deduce that the coefficients of each term in G(x) will be bounded by square root 10, or approximately 3.16, by definition of G2(x) and G(x) being automorphic. This process can be repeated for G3(x), where all coefficients are cubes of G(x) and the two functions are automorphic, and so on and so forth for G4, G5 etc. Effectively, as we increment the integer that we use to conduct the functoriality shift on G(x), we are increasingly restricting the bounds of the coefficients of G(x). G2(x) places bounds of square root 10, G3(x) will place bounds of cube root 10. Generalising, we Gk(x) will place bounds of 10 to the power of 1/k on the coefficients of G(x), until we eventually reach 1, thus proving the Ramujan conjectures. For a conjecture that still remains unsolvable, this is an extremely sharp and clever proof, considering its relative simplicity. Thus, we have demonstrated just how consequential and remarkable the Langlands Program is, and by extension the full modularity theorem and FLT, within mathematics.

How does this relate to the Mathematical Significance of FLT?

As part of Wiles's proof of FLT, he managed to devise a method to prove the Taniyama-Shimura Conjecture, or also known as the modularity theorem, for a special case. The modularity theorem states that all elliptic curves are modular forms, linking an analytic side of number theory to an algebraic side, respectively. He proved that semistable elliptic curves over a field 'Q' are modular. The field Q, here, refers to the set of rational numbers. A lot of highly advanced terminology has been used here ; it is not necessary to understand them to get a grasp of the significance of Wiles's proof of FLT. Rather, as we will explore later, it is much more important to know how novel and unorthodox Wiles's method of proving the theorem was, and how this has had major knock-on effects for a broad spectrum of mathematicians. Despite only proving the modularity theorem for quite a niche and specific case, this proof was highly innovative, engendering and giving breath to new and unconventional methods at proving the full theorem, which took place in 2001 Crucially, methods used by Wiles, Frey, etc, were not confined to just modular forms and elliptic curves. His proof can essentially be viewed as a pioneer in renewed and novel attempts at other theorems and conjectures. The modularity theorem and its implications for the Langlands program exist as just one example of this. Similar, lesser-known cases exist like the ABC conjecture ; It is evident that Wiles's proof of Fermat's Last Theorem has been hugely influential, through giving rise to fresh, unorthodox attempts at unresolved yet highly consequential conjectures.

Historical Significance

Ultimately, on top of its mathematical significance, the historical context and collaboration across lifetimes that surround FLT is what I believe makes it so meaningful as a mathematical proof. Throughout this essay, I have referred to the significance of FLT as a pioneer in the drive for more ambitious and consequential mathematical proofs, along with the new methodology that Wiles has produced. The main justification that gives FLT such influence is this historical context that we have been alluding to.

Fermat's Individual Legacy

It is first useful to understand Fermat's importance and legacy as a mathematician in order to appreciate the full context of FLT. He is best known for his early developments into rates of change on an infinitesimal scale (calculus), categorised by his own technique to calculate maxima and minima of functions, tangents to curves and other problems in calculus - adequality. At the time, this was very much a nascent field of mathematics. He made significant contributions to the field of number theory on top of this, with his Last Theorem being one of many. In the margin of his personal copy of Diophantus's Arithmetica, Fermat wrote his famous conjecture. Alongside the conjecture, he also stated ' I have assuredly found an admirable proof of this, but the margin is too narrow to contain it' (Fey, 1989, pg. 79). He was unable to produce and share an acceptable proof of his theorem within his lifetime, however. Thus, FLT is thought by many to make up a large part of Fermat's legacy as 'one of the greatest mathematicians ever' (Mielke, 1989).

The Journey Towards the Proof and the Importance of Collaboration

As such, despite the theorem only being officialised in around 1637, consistent and frequent attempts of a proof from a wide range of high-level mathematicians continued until its eventual proof in 1994 by none other than Wiles! Much of Wiles's eventual proof of FLT must be attributed to Taniyama and Shimura, who developed the Modularity/ Taniyama-Shimura conjecture in 1954. The conjecture hypothesised that all elliptic curves over the field of rational numbers are modular forms; it was heavily focused on algebraic geometry. As such, it initially seemed disparate from FLT, which appeared to be much more number theory-oriented. It took the work of many mathematicians to begin linking this theorem to FLT. Both Frey and Ribet contributed hugely to the eventual proof that the modularity conjecture would imply FLT. Frey linked the two mathematical statements together like so : Fermat 's theorem states that there are no solutions to the equation:

$$a^n + b^n = c^n$$

Figure 5 : Fermat's Last Theorem

Thus, consider the elliptic curve given by:

$$y^2 = (x)(x - a^n)(x - b^n)$$

Figure 6 : A special case of an elliptic curve

The general form of an Elliptic curve is given by y^2 = x^3 + Ax + B, where A and B are constants. Expanded out, Figure 5 would meet the requirements of the general elliptic curve form: there is no x^2 term. If the Taniyama-Shimura conjecture was proven to be true, then all elliptic curves are modular. If the special case of an elliptic curve given by Figure 5 was not modular, and Taniyama's conjecture was true, then the curve must not exist and FLT must be true. Ribet would go on to complete the linkage between the equation of the special elliptic curve in Figure 5, and the algebraic equation of FLT by proving two of Serre's Theorems. This would finalise the link between the Taniyama-Shimura conjecture and FLT. Wiles eventually proved the Taniyama-Shimura conjecture for semistable elliptic curves, which he realised was enough to imply FLT. Following countless, narrowly failing attempts at the proof from Miyaoka, Germain, etc, Wiles would make the monumental announcement of his proof during the final lecture in a series he gave at Cambridge University.

Ultimately, the proof was the culmination of the efforts of many hugely influential mathematicians like Fermat, Euler, Legendre, Dirichlet, Kummer, Faltings, Germain, Miyaoka, Frey, Taniyama, Shimura, Wiles, etc, to name a few. FLT was akin to a jigsaw puzzle, with each of these mathematicians, over 350 years of effort, gradually laying in the puzzle pieces, each one building on the work of the last. We would not be able to go into the detail that is deserving of each one in just this essay. However, with this alone, it is possible to get a feel for just how crucial collaboration was in the eventual proof of FLT, as well as how meaningful the proof was from a historical standpoint.

Conclusion

Fundamentally, examining the rich history of Fermat's Last Theorem and its impact in different mathematical fields has demonstrated the importance of collaboration in mathematical research to foster advancements that are both historically and mathematically meaningful. Using the analogy of mathematical proofs as a jigsaw puzzle again, each extra jigsaw piece added can provide more insight into the comprehensive puzzle. In the case of FLT, these initial jigsaw pieces were provided by the likes of Kummer, Frey, Tanimiya, etc. Areas of maths that may seem irrelevant, such as algebraic geometry and elliptic curves, actually played a huge part in the proof of FLT. This is especially prevalent in mathematics, which can be thought of as a tree with many interconnected nodes, all linked together. The vast interconnections prevalent in various fields of maths render localised and solitary attempts at proofs inefficient; instead, it is often highly beneficial to be provided with new insights from different fields and perspectives. This is what makes collaboration so essential!

Reference list

Boston, N. (2003). *THE PROOF OF FERMAT'S LAST THEOREM*. [online] Available at: https://people.math.wisc.edu/~nboston/869.pdf. [Accessed 4 Mar. 2024].

Brilliant.org. (2016). *Fermat's Last Theorem | Brilliant Math & Science Wiki*. [online] Available at: https://brilliant.org/wiki/fermats-last-theorem/. [Accessed 4 Mar. 2024].

Byerley, C. (2006). *Applications of Number Theory to Fermat's Last Theorem*. [online] Available at: https://www.whitman.edu/documents/Academics/Mathematics/byerleco.pdf. [Accessed 7 Mar. 2024].

Digitalcommons@usu, D. and Brown, A. (1996). *History of Fermat's Last Theorem History of Fermat's Last Theorem*. [online] Available at: https://digitalcommons.usu.edu/cgi/viewcontent.cgi?article=1300&context=honors#:~:text=On%20June%2023%2C%201993%2C%20Andrew. [Accessed 8 Mar. 2024].

Fey, J. (1989). Fermat's last theorem. Historical Topics for the Mathematics Classroom. VA: National Council of Teachers of Mathematics . Pgs. 79-80. [Accessed 8 Mar. 2024].

Koblitz, N. (1982). *Number Theory Related to Fermat's Last Theorem*. *Progress in mathematics*. Birkhäuser. doi:https://doi.org/10.1007/978-1-4899-6699-5.

[Accessed 7 Mar. 2024].

Mielke, P.T . (1989). Pierre de Fermat. Historical Topics for the Mathematics Classroom. VA: National Council of Teachers of Mathematics . Pgs. 410-3. [Accessed 8 Mar. 2024].

Quanta Magazine. (2022). *What Is the Langlands Program? | Quanta Magazine*. [online] Available at: https://www.quantamagazine.org/what-is-the-langlands-program-20220601/.

[Accessed 16 Mar. 2024].

wild.maths.org. (n.d.). *Andrew Wiles and Fermat's last theorem | wild.maths.org*. [online] Available at: https://wild.maths.org/andrew-wiles-and-fermats-last-theorem.

[Accessed 27 Feb. 2024].

www.maths.cam.ac.uk. (n.d.). *Fermat's Last Theorem - from history to new mathematics | Features: Faculty Insights*. [online] Available at: https://www.maths.cam.ac.uk/features/fermats-last-theorem-history-new-mathematics. [Accessed 10 Mar. 2024].

www.youtube.com. (n.d.). *Andrew Wiles: Fermat's Last theorem: abelian and non-abelian approaches*. [online] Available at: https://www.youtube.com/watch?v=4t1mgEBx1nQ. [Accessed 19 Feb. 2024].

www.youtube.com. (n.d.). *Elliptic Curves and Modular Forms | The Proof of Fermat's Last Theorem*. [online] Available at: https://www.youtube.com/watch?v=grzFM5XciAY. [Accessed 21 Feb. 2024].