How fast should I walk to school? Calculating my speed based on the rate of cooling of my morning coffee.

Introduction

I am a heavy caffeine addict. Every morning, I carry a paper cup with brewed coffee to school and I take sips of the coffee during my walk. I aim to finish my coffee before I face the hectic school life, whereas the temperature of my coffee does not always cool to the optimum temperature in my journey. Therefore, I always want to model the temperature change of my coffee to calculate an appropriate walking speed to enjoy the temperature at its best state every day. In this essay, I will model the temperature change of my coffee to work out an appropriate walking speed.

Theory

I researched that the ideal temperature of coffee for drinking is between 49℃~60℃[1], thus my coffee must reach the temperature range during my walk. To complete the model, I will relate the decreasing temperature of my coffee with the outside temperature based on the equation given by Newton's law of cooling, $Q = hA(T - T_{env})$, where Q is the rate of heat transfer out of the body, h is the heat transfer coefficient, A is the surface area, T is the temperature of the object's surface, and T_{env} is the temperature of the environment[2].

[1] https://www.homegrounds.co/uk/how-hot-should-coffee-be/

[2] https://knowledge.carolina.com/discipline/physical-science/physics/newtons-law-of-cooling/

Experiment

I then subsequently carried out some experiments and recorded the results. My first experiment measured the temperature of the coffee every 30 seconds, and I would note down the outside temperature. The second experiment repeats the procedure, but with coffee of a smaller surface area. In both cases, I would find a function that best describes the temperature change.

I have also collected data relating to the experiment: I measured the distance from my house to the school, and the dimensions (radius and height) of the paper cup with the coffee. Listed in the following table:

Distance from house to school	490m
Biggest radius of the coffee cup	3.45cm
Smallest radius of the coffee cup	2.70cm
Height of the liquid in the cup	6.40cm

Figure 1 Distance from house to school

Figure 2 Dimensions of the cup

Mathematical Manipulation of Newton's Law of Cooling

As mentioned before, Newton's Law of Cooling states that $Q = hA(T - T_{env})$. In the following, I would use $\frac{dT}{dt}$ instead of Q to represent the change of temperature over the change in time, and a coefficient α as a constant instead of hA because the heat transfer coefficient stays the same, and I would like to keep the surface area constant to start with for simplicity. Therefore, a differential equation $\frac{\mathrm{dT}}{\mathrm{dt}} = \alpha(T - T_{env})$ can be set.

Using separation of variables to solve the differential equation.

$$\frac{\mathrm{dT}}{\mathrm{dt}} = \alpha(T - T_{env})$$

$$\int \alpha\, dt = \int \frac{1}{(T - T_{env})} dT$$

$$\alpha t + c = ln|T - T_{env}|$$

The temperature of the coffee T is always higher than the outside environment T_{env}, so $T - T_{env} > 0$.

$$e^{\alpha t + \alpha c} = T - T_{env}$$

$$T = e^{\alpha t + \alpha c} + T_{env}$$

Let $e^{\alpha c} = A$.

$$T = Ae^{\alpha t} + T_{env}$$

Collecting Data through Experiments

Then, I experimented to collect data about the temperature of the coffee as it cools. The outside temperature, in this case is 18°C. This outside temperature is assumed in the following calculations as a constant. I use a thermometer to measure the temperature and note down the number every 30 seconds, which provides the following results.[3]

[3] The data of time 0-1200s is shown here, the full table with time 0-2400s can be found in appendix.

Time (secs)	Temperature of the coffee (°C)	Time (secs)	Temperature of the coffee (°C)
0	86	630	59
30	85	660	59
60	83	690	58
90	81	720	57
120	79	750	57
150	78	780	56
180	77	810	56
210	75	840	54
240	73	870	54
270	71	900	53
300	70	930	53
330	69	960	53
360	68	990	52
390	67	1020	52
420	66	1050	51
450	65	1080	50
480	64	1110	49
510	63	1140	49
540	63	1170	49
570	61	1200	48

Then, I plotted a graph (Figure 3) using the data above.

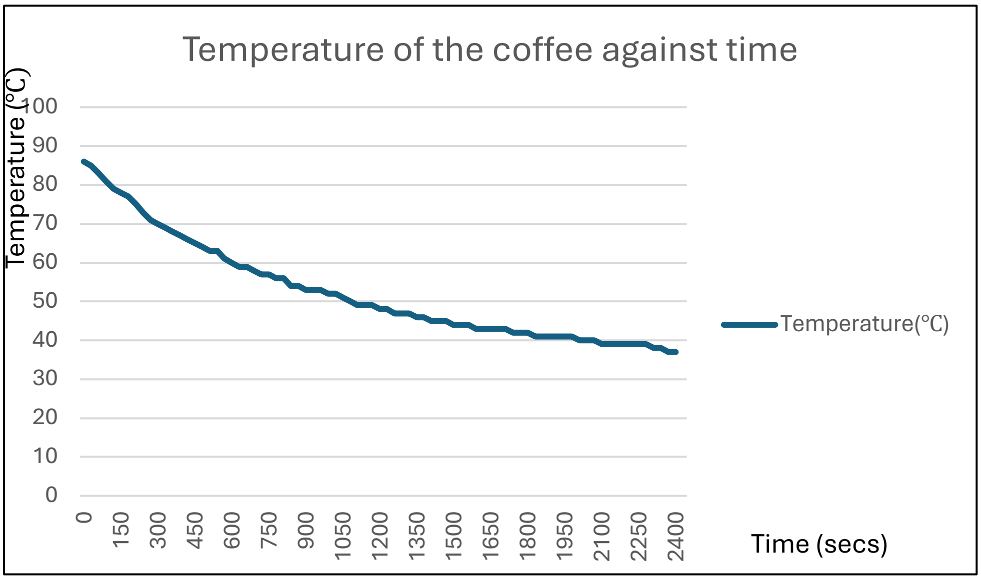

Figure 3 Temperature against time

The curve shows a distribution of exponential decay, which can support the previous assumption. To calculate the constant A, I chose arbitrary points on the curve. First, I take the point (0,86) (figure 4), which reflects the initial condition. This will give me the value of the coefficient A.

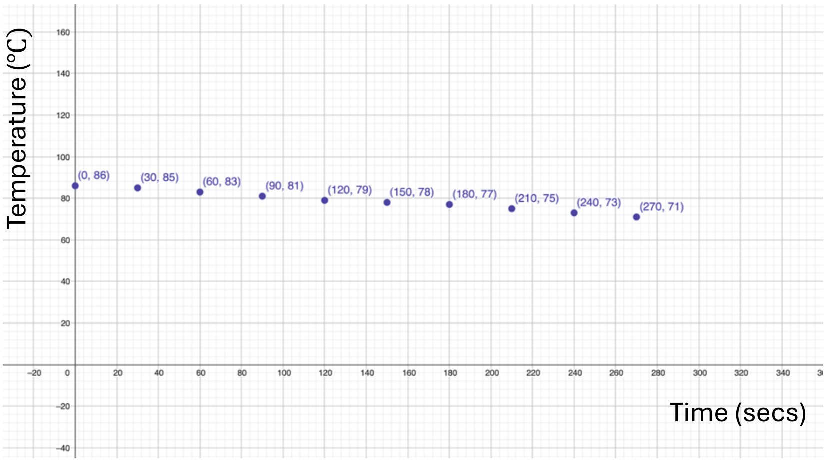

Figure 4 graph with the first 10 data points

Substitute t=0, T=86 into $T = Ae^{\alpha t} + T_{env}$.

$$86 = Ae^{0\alpha} + 18$$

$$\because e^{0\alpha} = 1$$

$$\therefore A = 86 - 18 = 68$$

$$T = 68e^{\alpha t} + T_{env}$$

Next, I chose the next random data point, (30, 85), to calculate α[4].

Substitute t=0, T=86 and t=30, T=85 into the equation to form a simultaneous equation.

Use A=68 attained from previous calculations.

$$\begin{cases} 86 = 68e^{0\alpha} + 18 \\ 85 = 68e^{30\alpha} + 18 \end{cases}$$

$$1 = 68 - 68e^{30\alpha}$$

$$\frac{67}{68} = e^{30\alpha}$$

$$ln(\frac{67}{68}) = 30\alpha$$

$$\alpha \approx -4.938 \times 10^{-4}$$

$$\therefore T = 68e^{-4.938\times10^{-4}t} + T_{env}$$

By plotting the curve on the graph (figure 5), it shows that the curve only fits the first four coordinates. This suggests that the points chosen are not representative of the curve: they are the first two data points, so they might reflect transient behaviours[5] when the change of temperature is first induced.

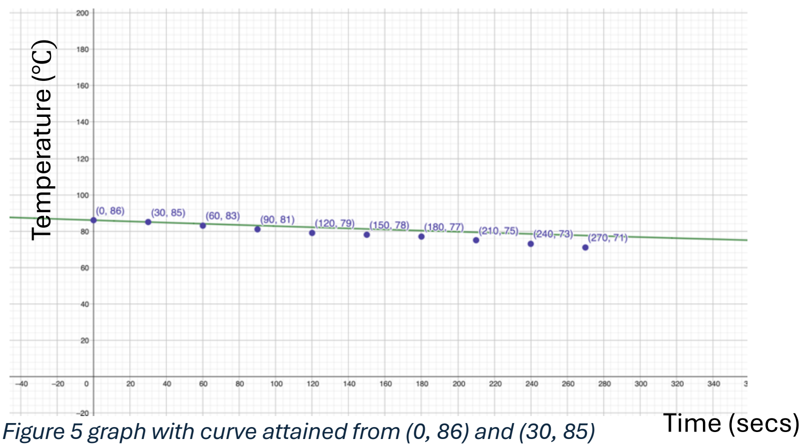

Figure 5 graph with curve attained from (0, 86) and (30, 85)

[4] α here, and all the following results in this project are rounded to 4 significant figures.
[5] The rate of convection and conduction had not match the temperature of the coffee yet.

Therefore, I realised that I must choose two points that are distant from each other on the curve, so they can be representative of the whole decay process. The resulting equation will then fit the most number of coordinates of the curve. Here, I chose the points (360, 68) and (900,53).

Substitute t=360, T=68 and t=900, T=53 into the equation to form a simultaneous equation.

$$\begin{cases} 68 = Ae^{360\alpha} + 18 \\ 53 = Ae^{900\alpha} + 18 \end{cases}$$

$$15 = A(e^{360\alpha} - e^{900\alpha})$$

Substitute $\frac{50}{e^{360\alpha}}$ into A.

$$15 = 50 - 50e^{540\alpha}$$

$$ln(\frac{35}{50}) = 50e^{540\alpha}$$

$$\alpha \approx -6.605 \times 10^{-4}$$

$$A \approx 63.42$$

Substitute the numbers into $T = Ae^{\alpha t} + T_{env}$.

$$\mathrm{T} = 63.42e^{(-6.605*10^{-4})t} + T_{env}$$

A second graph (Figure 6) is plotted below, which shows that the curve fits most of the points. However, this curve does not suit all the coordinates. A solution to this problem can be a piecewise function, where the first few data points are represented by one curve, and the rest by another. To testify which points do not suit the curve, I will calculate the percentage

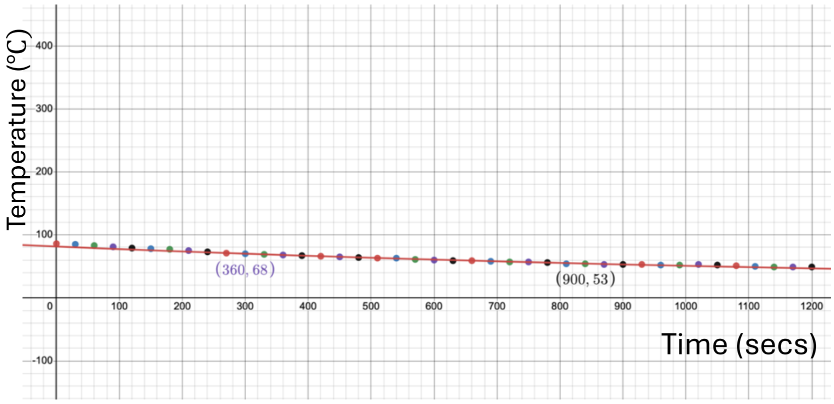

Figure 6 Curve with the equation attained from (360,68) and (900,53)

error of each coordinate. If there is a trend of percentage error being higher than 10%, I will construct a piecewise function with different domains.

Modelling a piecewise function

First, I calculated the percentage error of each coordinate and the expected value derived from the equation $T = 63.42e^{(-6.605*10^{-4})t} + T_{env}$. I discovered that, from t=1680s onwards, the percentage error exceeds 10% consistently.

Below is a table[6] of temperature and expected values from t=1560s, where an increasing trend of percentage error can be observed. For an accurate model, I think it would be the most suitable to add two more limits at $1680 \leq t \leq 2250$ and $2280 \leq t \leq 2400$[7].

Time (secs)	**Temperature (°C)**	**Expected Value (°C)**	**Percentage Error (%)**
1560	44	40.63	8.29
1590	43	40.19	7.00
1620	43	39.75	8.17
1650	43	39.33	9.34
1680	43	38.91	10.52
1710	43	38.50	11.69
1740	42	38.10	10.25
1770	42	37.70	11.40
1800	42	37.31	12.56
1830	41	36.94	11.00
1860	41	36.56	12.13
1890	41	36.20	13.26

[6] The full table is attached in the appendix.
[7] Here, I roughly divide the time periods by 10% of percentage error in the calculations: I will construct equations individually for the data that shows 0%-10%, 10%-20% and 20%+ of the percentage error attained from the equation $T = 63.42e^{(-6.605*10^{-4})t} + T_{env}$. This division lacks supporting proof due to the lack of a clear category for percentage error, for example, 20% can be accurate and inaccurate depending on the situation of the experiment.

1920	41	35.84	14.39
1950	41	35.49	15.52
1980	41	35.15	16.64
2010	40	34.81	14.90
2040	40	34.48	16.00
2070	40	34.16	17.10
2100	39	33.84	15.24
2130	39	33.53	16.31
2160	39	33.23	17.37
2190	39	32.93	18.44
2220	39	32.64	19.50
2250	39	32.35	20.56
2280	39	32.07	21.62
2310	38	31.79	19.53
2340	38	31.52	20.56
2370	37	31.26	18.38
2400	37	31.00	19.37

For the limit $1680 \leq t \leq 2250$, I picked the points (1680, 43) and (2160, 39). I use B and β to represent the new constants which are previously represented by A and α respectively.

Substitute (1680, 43) and (2100, 39) to form a simultaneous equation.

$$\begin{cases} 43 = Be^{1680\beta} + 18 \\ 39 = Be^{2160\beta} + 18 \end{cases}$$

$$4 = B(e^{1680\beta} - e^{2160\beta})$$

Substitute $B = \frac{25}{e^{1680\beta}}$ into the equation.

$$4 = 25 - 25e^{2160\beta}$$

$$\frac{21}{25} = e^{2160\beta}$$

$$ln\frac{21}{25} = 2160\beta$$

$$\beta \approx -8.072 \times 10^{-5}$$

$$B \approx 28.63$$

Substitute B and β into $T = Be^{\beta t} + T_{env}$.

$$T = 28.63e^{(-8.072\times10^{-5})t} + T_{env}$$

For the domain $2280 \leq t \leq 2400$, I picked the data points (2280, 39) and (2400, 37), using C and γ as coefficients.

Substitute (2280, 39) and (2400, 37) to form a simultaneous equation.

$$\begin{cases} 39 = Ce^{2280\gamma} + 18 \\ 37 = Ce^{2400\gamma} + 18 \end{cases}$$

$$2 = C(e^{2280\gamma} - e^{2400\gamma})$$

Substitute C= $\frac{21}{e^{2280\gamma}}$ into the equation.

$$2 = 21 - 21e^{2400\gamma}$$

$$\frac{19}{21} = e^{2400\gamma}$$

$$ln\frac{19}{21} = 2400\gamma$$

$$\gamma \approx -4.170 \times 10^{-5}$$

$$C \approx 23.09$$

Substitute B and β into $T = Ce^{\gamma t} + T_{env}$.

$$\mathrm{T} = 23.09e^{(-4.179\times10^{-5})t} + T_{env}$$

The percentage error of coordinates from t=1680 to t=2400 now are listed below.

Equation: T $= 28.63e^{(-8.072\times10^{-5})t} + T_{env}$

Time (secs)	**Temperature(°C)**	**Expected Value (°C)**	**Percentage Error (%)**
1680	43	46.53	7.59
1710	43	46.64	7.80
1740	42	46.64	9.95
1770	42	46.64	9.95
1800	42	46.64	9.95
1830	41	46.64	12.10
1860	41	46.64	12.10
1890	41	46.64	12.10
1920	41	46.64	12.10
1950	41	46.64	12.10
1980	41	46.64	12.10
2010	40	46.64	14.25
2040	40	46.64	14.25
2070	40	46.64	14.25
2100	39	46.65	16.39
2130	39	46.65	16.39
2160	39	46.65	16.39
2190	39	46.65	16.39
2220	39	46.65	16.39

Equation: T $= 23.09e^{(-4.179\times10^{-5})t} + T_{env}$

Time (secs)	**Temperature(°C)**	**Expected Value (°C)**	**Percentage Error (%)**
2250	39	41.05	5.00
2280	39	41.05	5.00
2310	38	41.05	7.44

2340	38	41.05	7.44
2370	37	41.05	9.88
2400	37	41.05	9.88

Given the percentage error has decreased, and there is no data point that reflect a percentage error of over 20% as before. I would like to draw the following conclusion:

For $0 \leq t \leq 1680$, the appropriate model is $\mathrm{T} = 63.42e^{(-6.605*10^{-4})t} + T_{env}$.

For $1680 \leq t \leq 2250$, the appropriate model is $\mathrm{T} = 28.63e^{(-8.072\times10^{-5})t} + T_{env}$.

For $2280 \leq t \leq 2400$, the appropriate model is $\mathrm{T} = 23.09e^{(-4.179\times10^{-5})t} + T_{env}$.

Calculating the Walking Speed

As mentioned before, the optimum temperature of coffee for drinking is between 49°C-60°C. Therefore, if I substitute T=49 and T=60 into the equation, an appropriate time period under a certain outside temperature (T_{env}) can then be calculated, which helps me calculate the suitable walking speed under the temperature. Here, I am going to use the same outside temperature (T_{env}) as above, which is 18°C.

Substitute T=49 and T=60 into $\mathrm{T} = 63.42e^{(-6.605*10^{-4})\mathrm{t}} + T_{env}$ respectively, where $T_{env} = 18$.

$$49 = 63.42e^{(-6.605*10^{-4})\mathrm{t}} + 18$$

$$\frac{31}{63.42} = e^{(-6.605*10^{-4})\mathrm{t}}$$

$$t \approx 1084$$

$$60 = 63.42e^{(-6.605*10^{-4})\mathrm{t}} + 18$$

$$\frac{42}{63.42} = e^{(-6.605*10^{-4})t}$$

$$t \approx 623.9$$

Thus, the coffee is at its optimum temperature between t=623.9 and t=1084, which gives me a window of 460.1 seconds. This is about 7.7 minutes to complete the journey to school while enjoying the uplifting drink at its best state.

The distance between my house to school is 490m, so the suitable walking speed is:

$$490 \div 460.1 \approx 1.065\ m/s$$

Therefore, I can conclude that when the outside temperature is 18°C, I need to walk at a speed of $1.065 m/s$ to school. Overall, I think this speed is very suitable as it corresponds with my walking speed normally: it usually takes me 6 minutes to walk to school, and this 7 minutes window will remind me to leave the house early in order to start my morning by relaxing and enjoying a cup of coffee on my way to school.

Also, as the coffee will not reach the optimum temperature after 623.9 seconds, which is 10.4 minutes, I think it would be best if I could leave it cool this time before my journey to school. This is a very perfect solution, as I can use the 10 minutes to toast my breakfast toast.

Appendix

Experiment 1: the change of the temperature of the coffee over time.

Time (secs)	**Temperature of the coffee (°C)**	**Time (secs)**	**Temperature of the coffee (°C)**	**Time (secs)**	**Temperature of the coffee (°C)**
0	86	840	54	1680	43
30	85	870	54	1710	43

60	83	900	53	1740	42
90	81	930	53	1770	42
120	79	960	53	1800	42
150	78	990	52	1830	41
180	77	1020	52	1860	41
210	75	1050	51	1890	41
240	73	1080	50	1920	41
270	71	1110	49	1950	41
300	70	1140	49	1980	41
330	69	1170	49	2010	40
360	68	1200	48	2040	40
390	67	1230	48	2070	40
420	66	1260	47	2100	39
450	65	1290	47	2130	39
480	64	1320	47	2160	39
510	63	1350	46	2190	39
540	63	1380	46	2220	39
570	61	1410	45	2250	39
600	60	1440	45	2280	39
630	59	1470	45	2310	38
660	59	1500	44	2340	38
690	58	1530	44	2370	37
720	57	1560	44	2400	37
750	57	1590	43	2430	37
780	56	1620	43		
810	56	1650	43		

Percentage error of all the data points, calculated from the equation

$T = 63.42e^{(-6.605*10^{-4})t} + T_{env}$.

Time (secs)	Temperature(℃)	Expected Value (℃)	Percentage Error (%)
0	86	81.42	5.625153525
30	85	80.1757013	6.017158091
60	83	78.95581571	5.122085375
90	81	77.75986424	4.166848527
120	79	76.58737731	3.150157082
150	78	75.43789454	3.396310934
180	77	74.31096459	3.61862536
210	75	73.20614498	2.450415911
240	73	72.12300189	1.215975603
270	71	71.06111005	-0.085996473
300	70	70.02005249	-0.028638215
330	69	68.99942046	0.000839922
360	68	67.9988132	0.001745327
390	67	67.01783783	-0.026616533
420	66	66.05610916	-0.08494167
450	65	65.11324959	-0.173927105
480	64	64.1888889	-0.294270396
510	63	63.28266414	-0.446669148
540	63	62.39421948	0.970892053
570	61	61.52320609	-0.850420711
600	60	60.66928195	-1.103164451
630	59	59.83211178	-1.390744468

660	59	59.01136687	-0.019262177
690	58	58.20672496	-0.355156482
720	57	57.41787009	-0.727770102
750	57	56.64449254	0.627611698
780	56	55.88628863	0.203469171
810	56	55.14296067	1.554213492
840	54	54.41421677	-0.761228956
870	54	53.69977082	0.559088385
900	53	52.99934227	0.00124101
930	53	52.31265612	1.313915091
960	53	51.63944272	2.634724938
990	52	50.97943776	2.001909562
1020	52	50.33238208	3.313210808
1050	51	49.69802161	2.619779117
1080	50	49.07610728	1.882571329
1110	49	48.46639488	1.10097959
1140	49	47.86864503	2.363457258
1170	49	47.28262302	3.632152512
1200	48	46.70809874	2.765904184
1230	48	46.14484662	4.020282914
1260	47	45.59264548	3.086801621
1290	47	45.05127852	4.325563109
1320	47	44.52053316	5.569265834
1350	46	44.00020101	4.544976943
1380	46	43.49007776	5.771252585
1410	45	42.98996312	4.675595727
1440	45	42.49966072	5.883198224

1470	45	42.01897803	7.094465668
1500	44	41.54772633	5.902305352
1530	44	41.08572057	7.093168607
1560	44	40.63277935	8.286956258
1590	43	40.18872483	6.995183815
1620	43	39.75338264	8.166895858
1650	43	39.32658186	9.340801998
1680	43	38.90815489	10.51667734
1710	43	38.49793745	11.69429544
1740	42	38.09576847	10.24846508
1770	42	37.70149003	11.40143258
1800	42	37.31494733	12.55543156
1830	41	36.93598858	11.00285
1860	41	36.56446499	12.13072585
1890	41	36.20023069	13.25894676
1920	41	35.84314265	14.38729132
1950	41	35.49306067	15.51553804
1980	41	35.14984729	16.64346551
2010	40	34.81336774	14.89839275
2040	40	34.48348992	15.99753996
2070	40	34.16008429	17.0957298
2100	39	33.84302387	15.23792953
2130	39	33.53218417	16.30617262
2160	39	33.22744313	17.37285907
2190	39	32.92868111	18.43778339
2220	39	32.63578078	19.50074141
2250	39	32.34862716	20.56153051

2280	39	32.06710748	21.61994975
2310	38	31.79111121	19.53026666
2340	38	31.52052997	20.55634861
2370	37	31.25525754	18.38008359
2400	37	30.99518974	19.37336183

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