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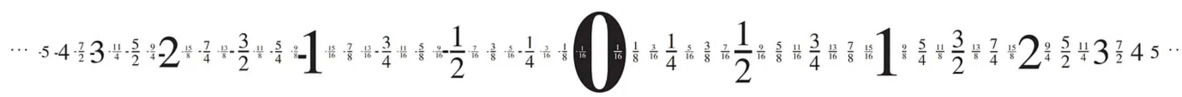
Surreal Numbers know no limits?
You *Knuth* be serious!

What does it mean for something to be surreal? In English, we define it to mean *a* bizarre or even freakish idea. Perhaps this is why Donald Knuth coined the phrase “Surreal Numbers” when describing John Conway's number system - Knuth saw this idea’s beauty and fantastical nature. He perhaps even saw it as a revelation, with frequent references to God in his writing, even naming Conway as J.H.W.H. Conway, likening him to Jehovah. Additionally, he states his book took 6 days to write (on the 7th he rested). Whilst *of course* there is an element of humour to this - a specialty of Knuth’s that’s clear when reading his book, the religious theme speaks to the fact that the concept of Genesis is mirrored in Conway’s “spiritual” idea.

Surreal numbers are a number system that works by creating new numbers from already defined ones. To put it simply, they are created by adding values between two pre-existing surreal numbers. It is something like the creation of the universe since the system builds on itself and a universe of numbers is created. Even within Knuth's book “Surreal Numbers”, he builds on this parallel by using the idea of the expanding universe as a metaphor for the ever-expanding field of surreal numbers.

As to how the numbers are described - a surreal number x will equal $\{ X_L : X_R \}$ where X_L and X_R are sets with created surreal numbers. $X_L < X_R$. This means the elements in X_L are not greater than or equal to the elements in X_R. Perhaps the most important number (everything else builds from it) is 0. So, on day 0 of using this system, we create 0 and define it as { : } which is a number bound by the two empty sets (and represents the idea of nothing). From this, we can easily create all numbers with denominator 2^n using the idea of subdivision. 1 is {0 : }, which is the integer between 0 and the empty set (of theoretical integers) greater than 0. The same principle applies in the opposite “direction” so -1 is { : 0}. **This idea carries on to give all integers n, with each n and -n being created on day n.**

From these integers, we continue using subdivision - averaging two numbers, to form the dyadic fractions. ½ is the number between 0 and 1, so ½ = {0 : 1}. -½ follows suit and is {-1 : 0}, formed on day 2 (alongside 2 and -2 since they require numbers from day 1 to be created). Since ¼ is between 0 and ½, we apply the same process to get {0 : ½} and then to get ¾ - the number between ½ and 1. Defining the dyadic fractions uses this idea of smaller and smaller 2^n fractions being the average between two 2^{n-1} fractions or a fraction and an integer, with **fractions base 2^n being created on day n+1.** Hence a number line emerges.



So far, this system has only given us integers and very specific fractions. While this is interesting, it isn't enough to make the surreal system largely useful. However, using the fact that **once we reach some infinite day ω we have created all the dyadic fractions and integers**, we can actually begin to create the reals! In this system, we define all the real numbers with two bounding (infinite) sequences. To define any real number ***a***, we represent it as { L_a : R_a } where L_a is a set of dyadic fractions less than ***a***, and R_a is a set of dyadic fractions greater than ***a***. Hence the fractions neighbour ***a*** on each side. Therefore, ***a*** emerges, as the value between these two sets. For example, we can define √2 as {1, 5/4, 11/8 ... : 3/2, 23/16…}.

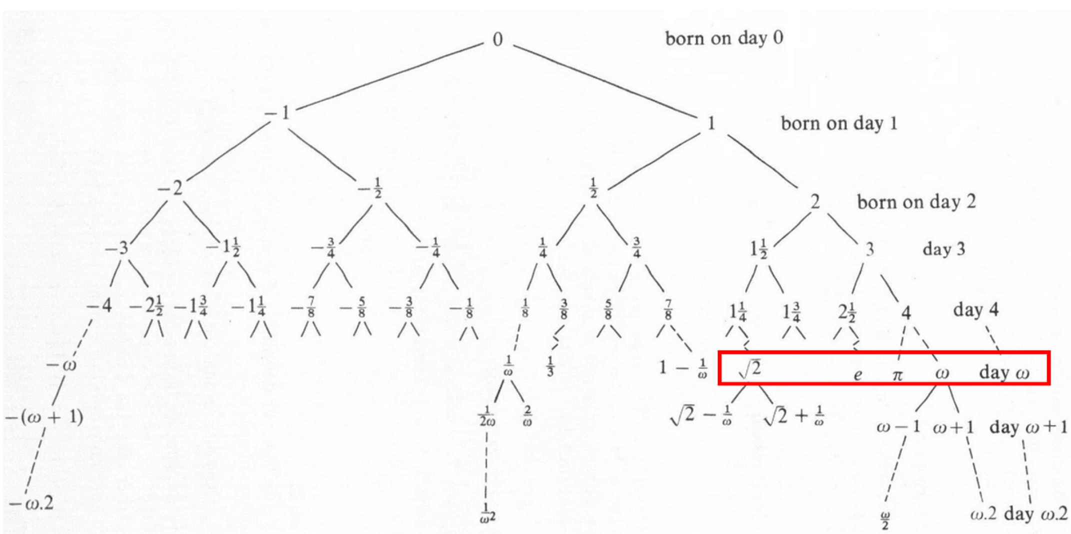
This idea of real numbers as compressing sequences of dyadic fractions is perhaps clearer when considering an example in binary. The numbers created after day ω are non-terminating decimals in binary. For example, the number ⅓ in binary is .01010… whereas for its surreal form in binary, its L_a is {.01, .0101, .010101, …}, approaching ⅓ from below and its R_a would be {.1, .011, .01011, …}, approaching ⅓ from above. Since these sequences are infinitely long, and a rule when creating surreal numbers is for a number x, all elements of X_L are less than X_R, L_a and R_a must terminate before they "crossover" or contain the same number. Therefore, ⅓ must be the number between the end of these two sequences. It is helpful to visualise this on a line.

.01, .0101, .010101, … x … .01011, .011, .1

This idea is different from convergence (as we will later explore why limits don't exist for the surreals) but relies on x being equidistant from X_L and X_R when they terminate. Similarly, other irrational values like π are defined as existing between two dyadic number sequences.

However, we aren't bound to just the reals, since we can use any two sets we like as the bounding sets. On day ω, we will create the number ω = { 1 2 3 4… : }. Since it has all the integers on the left, it is bigger than any integer. **Therefore, the number ω must be an infinity and is outside the set of reals.** Using this number, we can define very interesting values that don't exist in our real number system such as ω + 1, 5ω or ω^{ω} since ω is a number in this system and therefore we can use all operations with it. Another interesting result is 1/ω which we can call ε. Since ω is bigger than all the whole numbers ε must be less than all of them (as well as positive) and so ε = {0 : 1, ½, ⅓… }. **Since ε is less than all the positive real numbers, it is an infinitesimally small positive number**. Similarly, we can define - ω and - ε, which are infinitely large, and infinitesimally small negative numbers, that the surreal number system allows us to do arithmetic with.

Playing around with these numbers and evaluating what these results mean can be very fruitful, and I encourage you to do so. One that I find particularly interesting is to consider the number ω - 1 ≡ (1, 2, 3, 4,… : ω). It's the first number created that is larger than all integers - but at the same time, it's less than ω. We have *technically* created an infinitely large number that's less than infinity!



This idea of ε may seem interesting, but not necessarily that important. However, it actually undermines completely the idea of limits and therefore calculus within this system! This can be shown by thinking of a sequence of numbers such as 1/n. In our real number system, we see this to converge, as the limit for 1/n as n → ∞ is 0. This is intuitive as we know that as n becomes infinitely large, 1/n must become infinitely small. Using convergence definitions and formal logic in standard analysis (a sequence $\{ T_n \}$ converges to a real number ***a*** if for every x > 0, there exists an N ∊ Natural Numbers, such that whenever n > N, it follows that $| T_n - a | < x$), we can prove that 1/n converges to 0 as below:

Prove the $\lim_{n \to \infty} \frac{1}{n} = 0$

let $x > 0$ and $N \in \mathbb{N}$ with $N > \frac{1}{x}$ for all $n > N$.

$\left| \frac{1}{n} - 0 \right| = \left| \frac{1}{n} \right| = \frac{1}{n}$ (since $n \in \mathbb{N}$)

we want $\frac{1}{n} < x$

$\therefore n > \frac{1}{x}$

since $n > N$,

$N > \frac{1}{x}$ is the condition above

since $n > N$

$\frac{1}{n} < \frac{1}{N}$

$\Rightarrow \frac{1}{n} < \frac{1}{\frac{1}{x}}$

$\therefore \frac{1}{n} < x$

so we have proved formally the limit is 0

However, in the surreal system, this sequence does not converge. This seems intuitively true since we know that 1/n can take on infinitely many infinitesimally small values, such as ε, ε/2, and ε/1000000. To understand why this is actually the case, we can use a proof by contradiction as follows:

let $x = \varepsilon$ where $\varepsilon = \{0 : 1, \frac{1}{2}, \frac{1}{3} \dots\}$

$\Rightarrow$ ε is a number smaller than $\frac{1}{N}$ for all $N \in \mathbb{N}$

Since $n \in \mathbb{N}$ as it is the index of a sequence, for all n, $\varepsilon < \frac{1}{n}$

$\therefore \frac{1}{n} > x$

Therefore we have disproved the condition for 1/n to converge to 0. As we have shown, **limits don't have any meaning for the surreal number system**: 0.9999… no longer equals 1! This means that standard analysis breaks down within the surreal number system. Calculus has no meaning for the surreals since it is based on the behaviour of functions infinitesimally near a point, which limits allow us to study. The derivative, for example, is the limit that finds the slope of the tangent line to a function. That means for the surreal numbers we have to use a different form of analysis called non-standard analysis. It relates to non-standard calculus in which differentiation, continuity, and integrals are defined without limits.

So, what is non-standard analysis? Non-standard analysis is a branch of logic that uses something called hyperreal numbers. These are numbers that are greater than 0, but less than ½, ⅓, ¼… These are therefore essentially numbers of the form ε. It is analysis that defines calculus using infinitesimals (the hyperreals) in order to carry out formal logic. This is interesting as when learning calculus, the idea of limits seems deeply ingrained within it. But this idea of ε and the hyperreals bizarrely allows us to do calculus and analysis without limits - a *surreal* result indeed! **The number systems we use so greatly impact our conception of maths -** it is so fascinating to think of how the field would have evolved had we situated ourselves in the surreal world instead!