The Achilles' Heel of Mathematics– Aled von Oppell

Mathematics is a field which has changed society today. When you drive your car to school; this is because someone used mathematics to build the car, to work out multiple aspects such as gear ratios, required strengths as well as forces needed to overcome friction. Mathematics is also interlinked with many fields of study that are not immediately apparent. Mathematics can be seen as a strong pillar of foundation that holds our knowledge. For example, imagine you are baking a cake, and following a recipe using 100g of flour, six eggs, etc. This is using mathematics; how do you know what six eggs is? You had to use a numerical number line which indicates what you deem six to be.

Mathematics is the study of quality, structure, space, and change. Mathematics has shaped the world we call home entirely and changed sciences completely. The computer you are reading this from was made using mathematics. Nearly every subject has an aspect of mathematics in it, even if it just numeracy. The strong mathematics pillar is held up by axioms. An axiom is a principle that has been accepted to be true and correct and then used as a basis for argument in mathematics. However, our strong pillar of maths are in fact very weak unstable pillars that are supported by principles, that we do not know are completely correct.

A basic axiom that creates the groundwork for mathematical systems to work is described as a statement that is established as true but is not a theorem. A theorem is something that has been proven, whereas an axiom resembles a conjecture and is unproven until proven, when it then would become a theorem. However, an axiom cannot be proved, and is therefore a paradox!

A mathematical paradox is a mathematical conclusion so unexpected that is very difficult to accept, even though every step in the reasoning is valid. For example, a famous paradox is the Russell's Paradox which was discovered in spring of 1901, while Bertrand Russell was working on his Principle of Mathematics book. The paradox begins with his efforts to simplify set theory. The Comprehension Axiom tells us that for any set theory formula D where x is held as a free variable, their exists another set G whose members are exactly those objects that satisfy. The paradox now follows as if we let B stand for $x \in x$ and let $R = \{x: \sim\emptyset(x)\}$, then R is the set whose members are exactly those objects that are not members of themselves, therefore is R itself or is it not, and hence a paradox was created. Another famous paradox is the Berry Paradox which is a self-referential paradox which denotes to a statement that refers to itself or its own referent. Consider asking for "The first number that can't be described using 12 words or fewer" but the number we just described was described in twelve words creating a contradiction as we were able to describe it in twelve words or fewer. Therefore, a paradox is created. Mathematics has numerous paradoxes like these that lead to unexpected conclusions.

Gödel Incompleteness theorem is key proof of the existence of unprovable statements. Is the statement "This statement is False" true or false? If it is true then the statement is false meaning it is false, if its false then the statement is true. This is an unresolvable paradox; Gödel was fascinated by this, and he created the idea that perhaps this could also occur in mathematics. How would you say "this statement is false" in maths, as unlike in English the words have a way to communicate with each other. He created a new way to formulate equations into a single number. This way an equation such as "y=x" could be written as 1105 by using prime numbers to translate number so this by factorisation would be 13x5x17. This allowed him to create a way for maths to communicate more easily with each other. He was then able to show something is true, but it cannot be proven, therefore it is an unprovable statement which is true. This created a disruption in the community of maths as many mathematicians that thought every mathematical claim could be proven or disproven could not be. This created a sudden inconsistency where some ignored the claim Gödel made ignoring the massive change in the centre of the field, whereas some accepted, and others tried to get around this by making them axioms. The further increase in axioms of a problem, created an issue where they found more and more axioms which led some mathematicians to struggle.

Axioms hold up the entirety of mathematics as nearly all mathematical proofs used axioms, and many proofs also require assumptions. For example, in Physics assumptions are made that an object is made of particles, a string is inextensible, and a surface is smooth. We use assumptions to make equations easier to understand and therefore do not need to take into account other masses, friction as well as other factors. Without axioms, mathematics would not be able to work. These axioms and assumptions are the foundations of formal systems. Formal systems therefore have limitations due to axioms and assumptions. Many of the formal system limitations were highlighted in Gödel's work. One main limitation is the consistency; as a formal system is consistent it then cannot prove its own consistency, which means there are unprovable statements about its own consistency. In addition, formal systems are often designed with specific sets of axioms and rules of inference, limiting their ability to express certain mathematical concepts about parts of mathematics. Another example of its limitation is its model dependence. Formal systems are inherently tied to particular interpretations or models, and their conclusions may not hold in all mathematical contexts. These are only a few limitations of the formal system but there are more from Gödel's First Incompleteness Theorem as well as others. To try to limit the number of axioms and assumptions a full formal rigor is needed. But using full formal rigor to try to get over this hurdle of assumptions is difficult. Alfred North Whitehead and Bertrand Rusell attempted this. It took them over 300 pages to prove with formal rigor and show 1+1 = 2. They created further volumes and were going to create more. However, the formal rigor notation was too exhausting to complete more. Proving things with formal rigor would be exhausting and take longer then needed.

Gödel's Incompleteness further revolutionised the understanding of mathematical logic by showing these limitations of formal systems. Abraham Robinsons then developed non-standard analysis in the 1960s to address the limitation of traditional mathematical approaches. One of the famous parts of non-standard mathematics is Intuitionistic Mathematics, which is the principle that mathematics is a creation of the mind. The truth of a mathematical statement can only by made via a mental construction that proves it to be true. In Brouwer's original intuitionism, the truth of a mathematical statement is a subjective claim, he created a new form of logic called intuitionistic logic which derives from how mathematics is a subjective claim and that he didn't believe in mathematical realism. In addition, Constructive Mathematics is commonly held to be a branch of Intuitionistic mathematics that derives from Brouwer's intuitionistic logic. It is considered less with truth than with provability and it takes a more conservative view of what truth is than classical mathematics. But not all deductions of classical logic are considered valid. In classical logic to prove them, there exists an object having a certain property. It is enough to say no such object exists and derive a contradiction, while in constructive mathematics this isn't valid as you must construct an object having a certain property and prove it has the desired property.

Gödel's incompleteness theorem also created profound implications for philosophical mathematics. Gödel's theorems demonstrated that within any consistent formal system their exists statements that cannot be proven which challenged the notion of a complete and self-contained mathematics theorem thus raising questions about the limitations of human cognition and mathematical reasoning itself. This led to certain mathematical philosophical interpretations to become popular. Maths realism is one example and is the view that the truths of mathematics are objective, which is to say that they are true independent of any human activities, beliefs, or capacities. Therefore, humans did not invent mathematics, but discovered it. Kurt Gödel believed this, and that mathematics could be perceived in a manner analogous to sense perception. In addition, Maths fictionalism is also another philosophical interpretation. It was brought to fame in 1980 by Hartry Field. Maths fictionalism is the view that mathematical entities such as numbers, functions and sets are fictions that are useful for human purposes but are not themselves real in an ontological sense. Hartry Field showed how mathematics is a reliable process whose physical applications are also true.

In 1874, George Cantor opened a new branch of mathematics now known as set theory. Using the newly found set theory he found that there are different lengths of infinity. This was similar as infinity was something that cannot be described, something too great to be counted, yet he found a way to categorise infinities. He was given criticism and divided the mathematical community as intuitionists thought it was nonsense, but formalists agreed with him. It took years for the idea to be accepted. Now like Cantor, Gödel had found something that changed the entirety of mathematics due to it being the centre of mathematics, axioms that are used in every equation. Some mathematicians ignored the work and continue to try to hide the idea that this was true, others said it were wrong, whilst some accepted the idea. Gödel's theorem had changed mathematics and caused people such as the German mathematician David Hilbert who put forward a new proposal for the foundation of classical mathematics known as Hilbert's Program. The program started in the early 1920s then 10 years later had to stop due to Gödel's Incompleteness theorem. Gödel's Incompleteness theorems caused a ripple of effects in many parts of maths and in the end like Cantors' Set theory it was accepted.

Mathematics has a fundamental flaw as not all of mathematics can be proven and the structures that hold all of mathematics together are unprovable themselves. This then creates implications for mathematicians in what to do. To either accept mathematics has a flaw and continue, or to try find a way to prove it. And if mathematics basic elements are wrong then all of mathematics is wrong, as it is built on these unprovable statements. If they were wrong, it would not only cause problems in mathematics but also in nearly every other field of study such as physics, engineering, geography, economics and more, creating philosophical ideas and non-standard frameworks.

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