THE MANDELBROT SET IS UNIVERSAL

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1. Introduction

Many people have heard of and seen images of the Mandelbrot set, due to it's beautiful fractal nature. The Mandelbrot set arises from what at first seems like a chaotic, random set of points on the complex plane which come up when studying the border between chaos and order. This isn't a stand alone case, but is in a some sense "universal".

2. What is the Mandelbrot Set?

In a field of maths known as complex dynamical systems, one studies what occurs when you repeat the same function multiple times. In this case, the function at hand is $f_c(z) = z^2 + c$. The way we determine if a point c is in the Mandelbrot set is as follows: first, start with the input $z = 0$, then repeat the function many times at the point c and if it doesn't blow up then it is in the Mandelbrot set. For example, the point 1 is not in the Mandelbrot set because when we do this process with $c = 1$, we get:

$$0^2 + 1 = 1 \to 1^2 + 1 = 2 \to 2^2 + 1 = 5 \to 5^2 + 1 = 26 \to \dots$$

where clearly we see that this is going to diverge quickly. On the other hand, the point $c = -2$ will be in the Mandelbrot set because when we do the same process we obtain:

$$0^2 - 2 = -2 \to (-2)^2 - 2 = 2 \to 2^2 - 2 = 2 \to 2^2 - 2 = 2 \to \dots$$

which clearly doesn't explode. We can also do this process for complex numbers, for example the point $c = 0.328 + 0.117i$ also doesn't explode and it's "orbit" looks like this:

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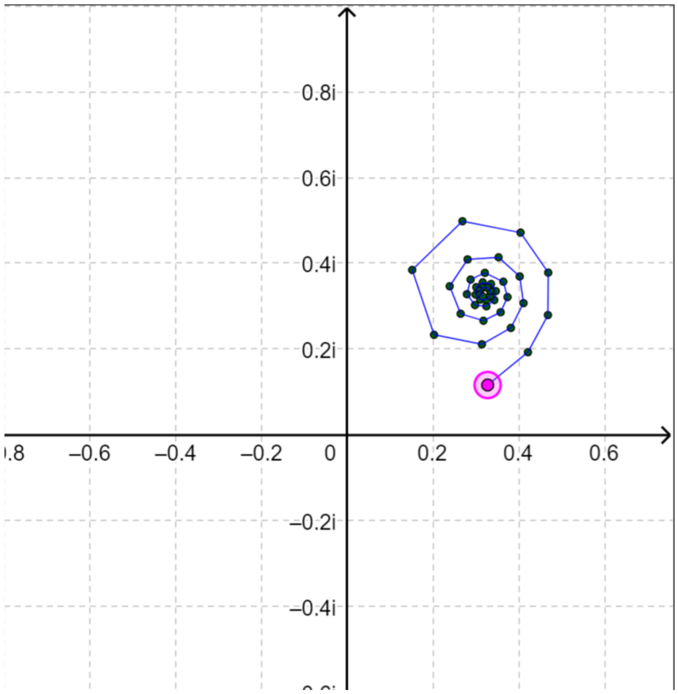

FIGURE 1. $c = 0.328 + 0.117i$

I encourage you to play around here on geogebra to see what to happens to specific points- at first there does not seem to be any obvious pattern. However, within the chaos, there lies great beauty: the Mandelbrot set

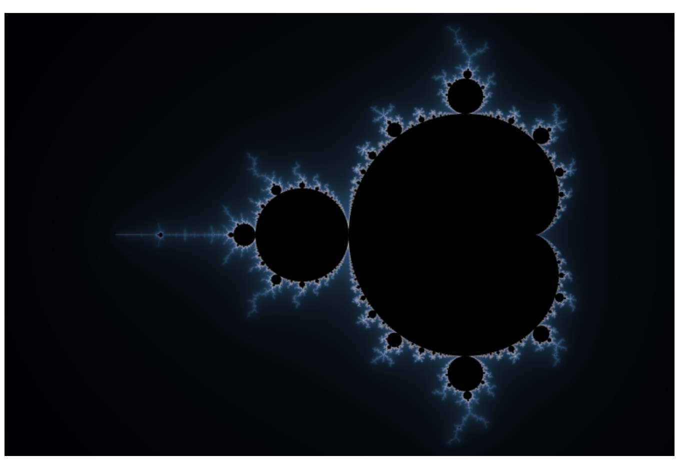

FIGURE 2. Mandelbrot Set: Image taken from fractalposter

3. The Newton Raphson Fractal

We now turn our attention to something which at first seems completely unrelated: the Newton Raphson method. The Newton-Raphson method is a way of approximating solutions to equations. For some simple equations like quadratics, one can use the quadratic formula. However, For cubics and quartics, the corresponding formulae will give you solutions, but they are not nearly so pleasant to work with. Moreover, there are no formulae for quintics or above, so we can't get the exact solutions in this way. Thus, we approximate it instead (and the same is done for quartics and cubics, due to their clunky formulae). For a polynomial $p(z)$, we guess an answer z_0 and then we iteratively use the recursive formula to get an increasingly accurate estimate:

$$z_{n+1} = z_n - \frac{p(z)}{p'(z)}.$$

For example, if we want to approximate the square root of 2, we can use the Newton-Raphson method on the equation $z^2 - 2$. If we guess $z_0 = 1.5$ then we have:

$$\begin{aligned} z_0 &= 1.5 \\ z_1 &= 1.5 - \frac{1.5^2 - 2}{2 \times 1.5} = 1.42 \\ z_2 &= z_1 - \frac{z_1^2 - 2}{2z_1} = 1.41 \\ z_3 &= \dots \end{aligned}$$

here we see how quickly the Newton-Raphson approximated $\sqrt{2}$, which shows the effectiveness in approximating roots. The question is: how is this related to anything we were talking about before? The answer lies within the fact that there are some starting points where the Newton-Raphson method doesn't work e.g. $z_0 = 0$ when finding $\sqrt{2}$

We are going to use this fact to play a game now: we consider the polynomial $f_\lambda(z) = (z+1)\left(z - \lambda - \frac{1}{2}\right)\left(z + \lambda - \frac{1}{2}\right)$. Now we will use the Newton-Raphson method to approximate the roots of this polynomial, always using the starting guess $z_0 = 0$. We colour the points according to which root they converge to $\left(\lambda + \frac{1}{2}; \frac{1}{2} - \lambda; -1\right)$ and here's the crucial point: we colour λ black if the Newton-Raphson method started at $z_0 = 0$ doesn't converge to any root! The result of this process is the following:

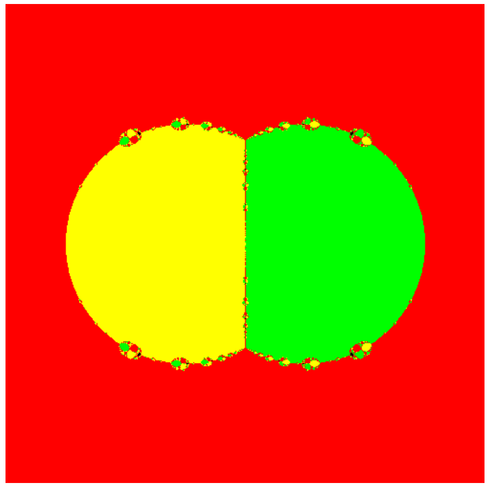

FIGURE 3.

Whilst this looks kind of boring, lest we forget that the motto for this essay is that: there is beauty hidden in the chaos, and lo and behold when we zoom into the point $\lambda = -0.19134 + 0.296738i$, we obtain the following image:

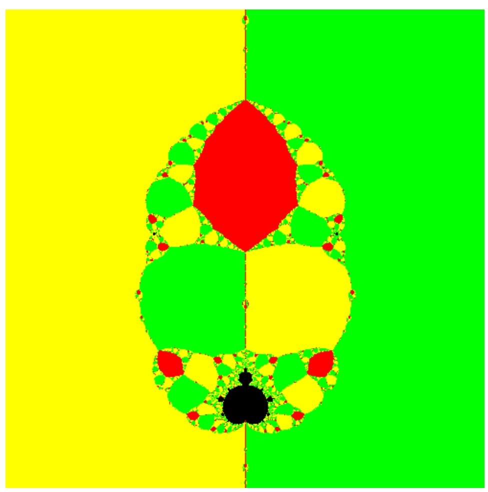

FIGURE 4.

Hidden, there lies a Mandelbrot set within the fractal! (In fact if you zoom in further there are infinitely many tiny little Mandelbrots).

4. Making Our Own "Hidden Mandelbrot" Fractals

Now we see the final result in its full glory- the occurance of the Mandelbrot set is indeed not a mere coincidence, but in fact a much more general occurance which we can use to make our own fractals with hidden Mandelbrots. The game goes as follows:

(1) Consider some rational function $f_c(z)$ (which means that it's in the form $\frac{g_c(z)}{h_c(z)}$ for some polynomials g, h.
(2) Then find the root(s) of $f_c'(z)$.
(3) Use the root(s) as a starting point and colour a point c in one colour if it diverges after iterating $f_c(z)$ a bunch of times and another if it converges.
(4) Observe.

We shall now finish by looking at plenty of examples, starting with $f_c(z) = \frac{z^4}{z^2+c}$. We shall follow the steps outlined above. Firstly, the derivative of $f_c(z)$ is simply $\frac{2z^3(z^2-2c)}{(z^2+c)^2}$ which is equal to zero at $z = 0$ and $z = \pm\sqrt{-2c}$. Now, clearly if we were to start out our function with $z = 0$ and iterate a bunch of times we would just get nowhere, and starting with $z = -\sqrt{-2c}$ will give the same thing as starting with $z = \sqrt{-2c}$. Therefore, we shall colour in a point c green if the polynomial converges aftering being iterated (starting with $z = \sqrt{-2c}$) and red if it diverges. It looks like this:

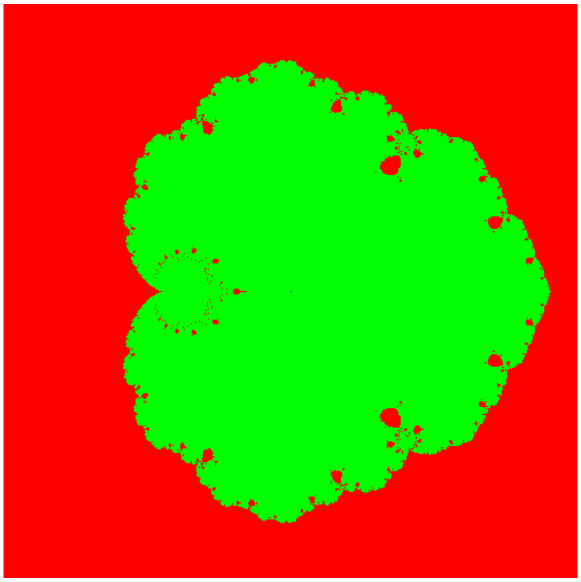

Figure 5.

lo and behold, when you zoom in closer you can clearly see a Mandelbrot

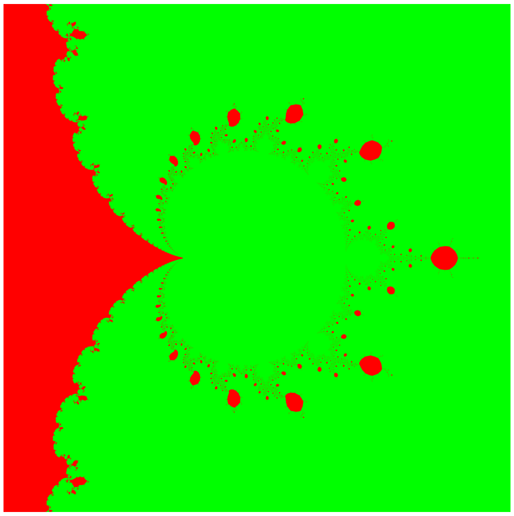

FIGURE 6.

Our next example is $f_c(z) = z^4 - \frac{c^2 z^2}{2}$, which is conveniently chosen so that the derivative is $z(z-c)(z+c)$, so we shall start with the point $z = c$ and again colour a point c red if iterating the function (starting at $z = c$) causes it to diverge and colour it green if the function converges to a point after being iterated with the starting point $z = c$. This time our fractal looks like this:

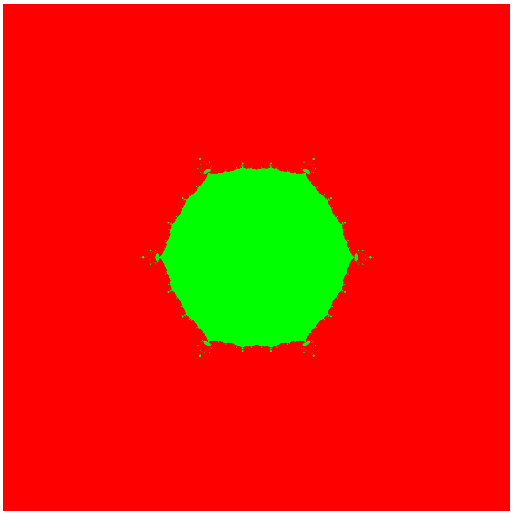

FIGURE 7.

Now if we zoom in on one of those "legs" on the side, we see that it looks like this:

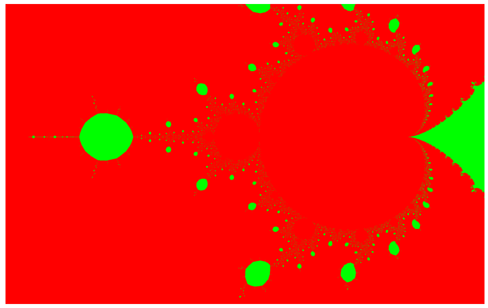

FIGURE 8.

For our final example, we need to clean up two issues: what happens if the derivative has more than one root, and does the Mandelbrot set always appear when we play this game? We answer this with one example: $f_c(z) = \frac{z^3}{3} - \frac{3z^2}{2} + 2z + c$, which has the derivative as $f_c'(z) = (z-1)(z-2)$. We now use the following colour code: if starting with both $z = 1$ and $z = 2$ at the point c converges, then we colour the point cyan. If only starting with $z = 1$ converges but starting with $z = 2$ diverges, then colour c blue, if starting with $z = 2$ converges but starting with $z = 1$ diverges, then colour the point c in green and if both of the starting points cause the function to diverge at the point c when iterated then colour c black. Now we get the following interesting fractal:

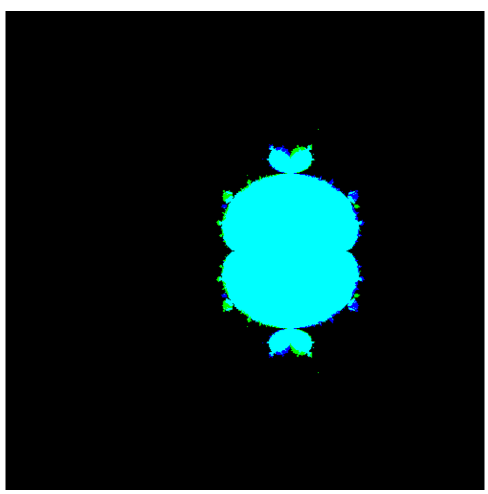

FIGURE 9.

This isn't the Mandelbrot set: what's happening? Well, the Mandelbrot set is part of a family of fractals known as the Multibrot sets,

M_n, which occur from doing the exact same process as we did for the Mandelbrot set but for the function $f_c(z) = z^n + c$. The fractal that we just obtained is M_3. And indeed, the Multibrot sets will always pop up if we do the process outlined in the final section of this paper, no matter what rational function you chose (so long as you don't pick one where it always converges/diverges). If you want to look into this phenomena in more precise detail then I have also previously made a video on the topic for you to watch.

5. Acknowledgements

I'd like to give a big thank you to Saik Rudad for his great help in wording this paper, to David Baah-Okyere for telling me about this competition and to Alexander Zatwardnicki and Vincent Karthaus for coding all of the beautiful hidden Mandelbrot images that you see in the second half of the paper.