Euler's Formula: Beauty in Simplicity

In 2014, research was conducted by presenting 16 different mathematicians with 60 different mathematical formulas. During this time, their brain activity was being captured by MRI and thus yielded unanticipated results. Researchers found that certain parts of the frontal lobe (an area stimulated when the brain processes visual or auditory beauty such as art or music) were particularly activated when a specific formula was shown: the Euler equation.

$$e^{i\pi} + 1 = 0$$

While there are preconceived notions by the general public that indicate that mathematicians and physicists gravitate toward complex formulas, the reality is quite the opposite. Mathematicians and physicists are likely to appreciate simplicity and symmetry more than most. This is due to a profound understanding of mathematics, thus leading to a deeper appreciation for the Euler equation.

5 protagonists

Prerequisites

1. '0': 0 is a number that represents an empty quantity. It is the identity source of addition for all algebraic structures (e.g. real/imaginary/complex numbers).
2. '1': 1 is a number representing real numbers. It is the identity source of multiplication for all algebraic structures (e.g. real/imaginary/complex numbers). 1 is also the radius of a unit circle (i.e. the sum of the squares of cosine and sine).
3. 'i': 'i', representing imaginary numbers, is the root of the equation: $x^2 + 1 = 0$. It is the square root of (-1). Even though there are no properties in real numbers, it is used to expand real numbers to complex numbers.
4. 'π(pi)': π is a constant that represents a circle. It is the ratio of a circle's circumference to its diameter with an approximate value of 3.14. When an angle is radiated in the unit circle, π also represents the circumference of the semicircle.
5. 'e': Euler number, also known as the mathematical constant, is a mysterious number that can be defined in two ways.

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n \qquad e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{1} + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \cdots$$

How do these two definitions have the same value when both are expressed in exponential limits and the sum of the reciprocal? What mathematical significance does this value have?

Imagine you've gone to a bank on a compound interest that gives 100% interest annually. A dollar kept in the bank will be added to one dollar of interest and become two dollars. Consider the money left in this bank account when withdrawing money in different cycles and save again. If you offer after 6 months, you'll get 50% interest. At the end of the year, a dollar will have become 2 dollars and 25 cents. This concept can be expressed through the equation $(1 + \frac{1}{2})^2 = 2.25$. If you do the same thing every 3 months, $(1 + \frac{1}{4})^2 = 2.44$ dollars will be left at the end of the year. Doing this every month, you'll get 2.61 dollars.

How much money can you get when going through this process every week, day, hour, or even second? Will it expand limitlessly? When this process is repeated on a limited cycle, the money generated at the end of the year is expressed as follows.

$$e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$$

This limit (i.e. Euler constant ‘e’), converges to an infinite decimal, 2.71828…

Euler’s number is also defined as an infinite sum of reciprocal factorials. How is this sum equal to the same value as the one prior? With a small change in perspective, calculations can be made for the sum of the deposit, the interest of the deposit, the interest of the interest of the deposit, and so on.

The initial deposit was 1, and the total interest accrued for a period of one year is 1. Therefore, the deposit’s function concerning time would be constant.

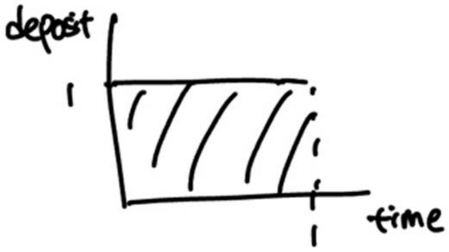


Figure 1: Deposit for time

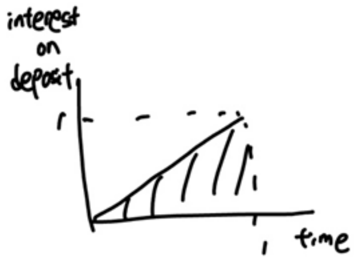


Figure 2: Interest of deposit for time

The interest on the original principal will rise in a straight line. This is because interest is a function that integrates deposit (a constant function) over time. The total interest on deposit for a year will be $\int_0^1 1\,dx = 1$. If this is the case, how would the second round of interest on the deposit be evaluated? Interest upon interest on a deposit is a function that integrates interest on a deposit over time. The total interest on interest on deposit will be $\int_0^1 x\,dx = \frac{1}{2}$. A certain rule can be found based on this interest.

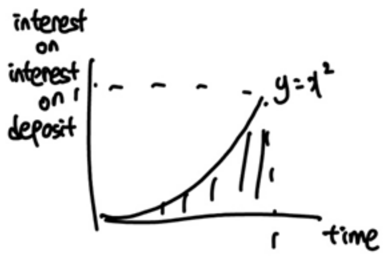


Figure 3: Interest on interest on deposit for time.

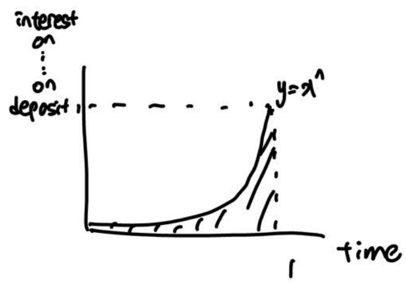


Figure 4: Interest on ... on interest on deposit for time.

In the same way, interest on interest on deposit for time will be evaluated as $\int_0^1 x\,dx = \frac{1}{6}$.

Repeating this n times, the interest on interest on … on deposit will be evaluated as $\int_0^1 \frac{x^n}{n}\,dx = \frac{1}{(n+1)!}$.

To calculate the total money, all the interest should be aggregated infinitely.

After aggregation, all of these will be expressed as

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = \frac{1}{1} + \frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \cdots$$

Now, all 5 constants used in Euler's equation, the most beautiful formula, have been explained. What seems fundamentally incomprehensible from the Euler equation is that there is an imaginary number in the exponent. Does $e^{i\pi}$ mean that e is multiplied $i\pi$ times? This expression itself is incoherent. Before attempting to understand the implications of having an imaginary number in the exponent, two properties of the complex number should be pointed out.

Real numbers, including integers; rational numbers; and irrational numbers can be expressed on a vertical line. Every real number is positive when it's squared. A number 'i' is defined as a square root of '-1', and all its multiples are imaginary numbers. Complex numbers, which are a linear combination of real numbers and imaginary numbers, require a two-dimensional space that consists of a real number axis and an imaginary number axis to be expressed. This two-dimensional space is called a complex plane.

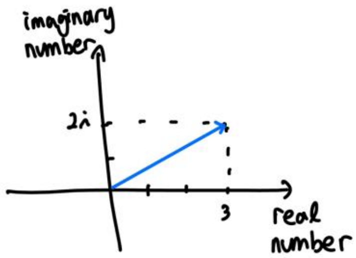


Figure 5: Complex plane.

In this plane, 3+2i can be expressed as the point above. Because 1 and i are linearly independent, a point in the complex plane can be expressed as a position vector: an arrow from the origin.

A complex plane has two important features. The first is regarding the addition of complex numbers. Complex numbers can be considered as a vector in a complex plane. A complex number is a linear combination of a real-axis direction unit vector and an imaginary-axis direction unit vector. For example, 3+2i is the sum of three times the real number direction unit vector and two times the imaginary number direction unit vector. The addition of a vector is determined by the sum of each component. Vector summation is the addition of each position vector. In the figure below, two vectors can be added by adding the endpoint of the red arrow and the start point of the blue arrow. As such, the addition of complex numbers is the same as vector combinations.

- For convenience, the real number axis will be the x-axis and the imaginary number axis will be the y-axis.

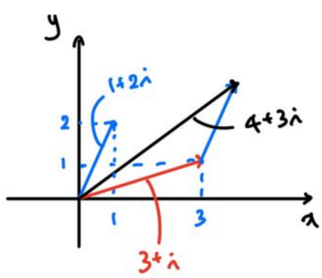


Figure 6: Summation of complex numbers.

The second feature regards multiplication. Multiplying i to a complex number reaches the point that rotates 90 degrees counterclockwise on the complex plane. For instance, think of a complex number z, 4+3i. Multiplying i to z will result in the following equation by the distribution law of the product. It is calculated the same way as real numbers, except that i square is -1. -3+4i, 4+3i multiplied by i, is a point where 4+3i is rotated 90 degrees counterclockwise. This can easily be proven through the Pythagorean Theorem for any complex numbers.

$$\begin{aligned} z \times i &= (4+3i) \times i \\ &= 4 \times i + 3 \times (-1) \\ &= -3 + 4i \end{aligned}$$

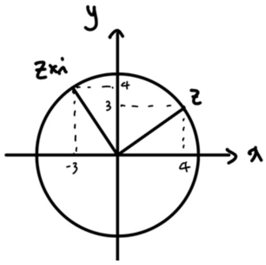


Figure 7: Distribution law **Figure 8: Multiplication of complex numbers.**

To sum up, there are two components to remember regarding complex planes.

1. The sum of complex numbers is calculated as a vector addition.
2. Multiplying 'i' to a complex number is rotating 90 degrees counterclockwise on the complex plane.

Moving on to understanding Euler's equation; differential equations should be utilized. Building and solving this differential equation is a universal way of analyzing natural phenomena which is eminently important. Newton's universal gravitational force, electromagnetic force, spring resilience, wave equation, and diffusion equation are all examples of instances in which differential equations are used.

To solve differential equations, start with basic derivatives. It is commonly known that differentiation is the limit to a point of slope. A slope is the amount of change in x relative to the amount of change in y. The value of differentiation at a point is the slope between two adjacent points, in which x coordinates differ by Δx; a very small gap. It is also the slope of the tangent line at a point in the function.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

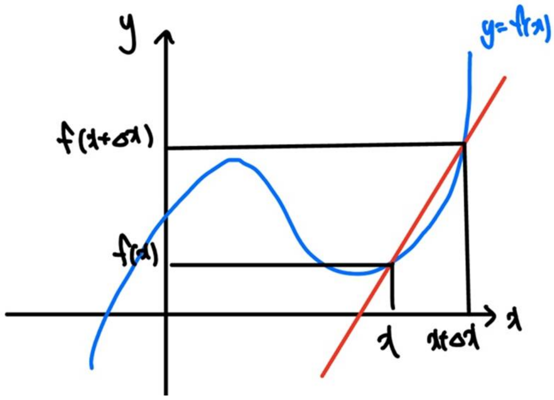


Figure 9: Derivative definition **Figure 10: Derivative in graph.**

To simplify the equation, erase the intercept limit symbol for a moment, and keep in mind that Δx is a minuscule value. After multiplying both sides by Δx and transporting the above differential equation, the follox wing equation is created.

$$f(x+\Delta x) = f(x) + f'(x)\Delta x$$

Now, look at the case where the function f(x) satisfies some of the simplest conditions; f(x) equals f'(x).

$$f'(x) = f(x)$$

The random function discussed to draw a graph does not satisfy the condition above. In layman's terms, there is a point in which the function value is positive, but the slope is negative.

Any point should first be used to detect what kind of function the condition above satisfies. Suppose it passes through (0,1). This is to create the simplest form in which both the slope and the function value are 1 at x=0. Adjacent points starting from (0,1) can be drawn based on the condition that the function value and the slope are the same.

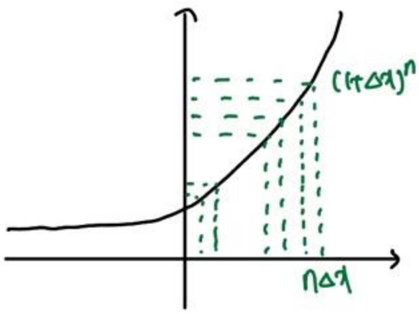
Using this condition, the equation above can be transformed as follows.

$$f(x+\Delta x) = (1+\Delta x)f(x)$$

By sequentially substituting 0, Δx, and $2\Delta x$ for x, some kind of regularity can be found in the function values. Each increase by Δx increases the function value by a factor of (1+ Δx). The function values form an equal sequence with a covariance of (1+ Δx).

$$\begin{aligned} x=0 &\quad f(\Delta x) = (1+\Delta x) \\ x=\Delta x &\quad f(2\Delta x) = (1+\Delta x)^2 \\ x=2\Delta x &\quad f(3\Delta x) = (1+\Delta x)^3 \end{aligned}$$

Inductively, since this sequence has a covariance of (1+Δx), the nth term will have a value of (1+$\Delta x)^n$. Sending n to infinitely large numbers and Δx towards small intervals, the graph's points on the coordinate plane will trace a continuous exponential graph.



Replacing nΔx as x, f(x) can be obtained as follows.

$$f(x) = \left(1+\frac{x}{n}\right)^n$$

Don't forget that Δx was a small value. By sending it infinitely small, the final form of the function satisfying the above differential equation can be expressed as follows.

$$f(x) = \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$$

This equation may seem familiar. When substituting x=1, the function value equals the Euler constant. We can discover that the function is e^x. This function is both an nth-order polynomial and an infinite-dimensional polynomial.

Collecting all the essential constants is just as necessary as Thanos collecting all the infinity stones. We met Euler's constant and number 1 through the most basic differential equation. Number 1 is hidden in the coefficient of the differential equation.

$$f'(x) = 1 \times f(x)$$

Just as we met Euler's constant in the differential equation for 1 representing real numbers, meeting more constants can anticipated by solving the differential equation for ‘i’ that represents imaginary numbers.

$$f'(x) = i \times f(x)$$

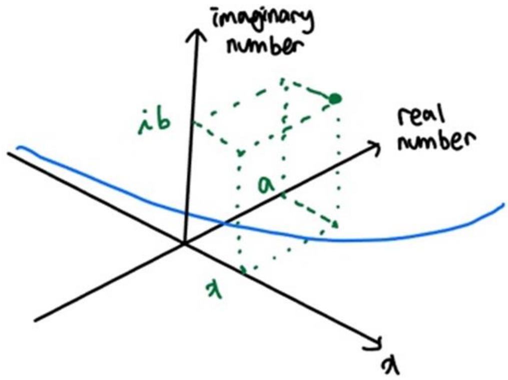
This differential equation may look difficult, however, once the solution process of the previous equation is taken into account, it can be solved simply. It is processed in the same sequence as the previous.

$$\begin{array}{ll}
f'(x) = 1 \times f(x) & f'(x) = i \times f(x) \\
f'(x+\Delta x) = (1+\Delta x)\, f(x) & f'(x+\Delta x) = (1+i\Delta x)\, f(x) \\
f'(\Delta x) = (1+\Delta x) & f'(\Delta x) = (1+i\Delta x) \\
f'(2\Delta x) = (1+\Delta x)^2 & f'(2\Delta x) = (1+i\Delta x)^2 \\
f'(3\Delta x) = (1+\Delta x)^3 & f'(3\Delta x) = (1+i\Delta x)^3 \\
\vdots & \vdots \\
f'(n\Delta x) = (1+\Delta x)^n & f'(n\Delta x) = (1+i\Delta x)^n \\
e^x = \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n & e^{ix} = \lim_{n\to\infty} \left(1+\frac{ix}{n}\right)^n
\end{array}$$

If if(x) is substituted instead of f(x) to f'(x) at the beginning of the solution, only the covariance is changed to (1+iΔx). Therefore, e^{ix} is expressed as a polynomial.

For $e^{i\pi}$, you can just substitute π for x as the polynomial. However, it is difficult to recognize the value in this way. Multiplying e by $i\pi$ times is irrational. However, if $e^{i\pi}$ is expressed in the form of an infinite-order polynomial as shown above, the calculation is possible. This value is obtained by multiplying $(1+\frac{i\pi}{n})$ n times for an infinitely large number n. This calculation is an arithmetic operation for a complex number.

First, e^i is seen by substituting 1 for x. Because $e^{i\pi}$ is a complex number, it will have a real number part and a complex number part. Knowing where this value is in the x-squared graph of e drawn earlier is necessary. If x is a real number, then for any x, e^x is a real number, so this can correspond to the y-axis containing all real numbers. The real number part corresponds to a point on the y-axis for the e^i. However, to express the imaginary part, it cannot be expressed in a two-dimensional plane. The imaginary axis is z-direction perpendicular to both x and y.



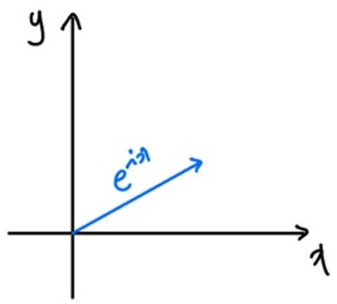
The coordinates of e^i and a+ bi, in a three-dimensional space, are the variable x value of 1, the real number negative y value of a, and the imaginary number z value of ib. The only point yet known is (0,1,0) when x=0; real part =1 and imaginary part = 0. As x changes, the x-square of e will travel along some path in a three-dimensional space passing through (0,1,0).

Since analysis of three-dimensional space is difficult, projecting it on the yz plane yields a graph projected on the complex plane. Knowing what a projected graph implies is necessary. All that is known is that it passes (0,1,0) when x=0. When x becomes any value x=a, the real part of e^{ia} becomes the y-coordinate and the imaginary part becomes the z-coordinate. As x continues to grow, the location of the (ix) power of e will continue to change, and the trace of these values is the green curve.

The graph e^{ix} can be drawn in the same way as the method of drawing the graph e^x earlier. Like what we did earlier, we will photograph the values that change as Δx increases on the complex plane. $e^{ix'}$s value adjusts as follows the equation below.

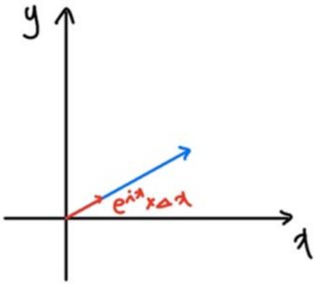
$$e^{i(x+\Delta x)} = e^{ix} + ie^{ix}\Delta x$$

Using the two rules of operation in the complex plane discussed above, it is not difficult to photograph points of e^{ix}.

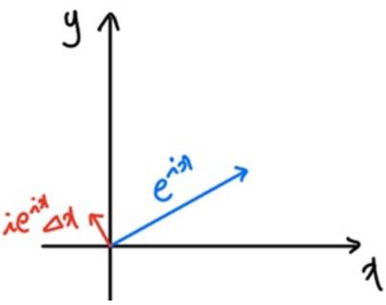


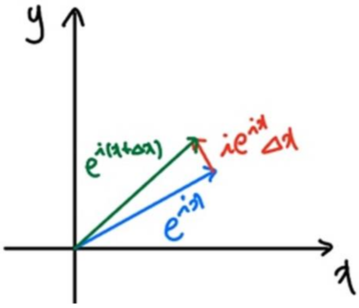
Suppose that e^{ix} exists somewhere in the complex plane. How does the term added to e^{ix},

e^{ix} multiplied by i and Δx , appear on the complex plane? Foremost, $\Delta x e^{ix}$is the same direction vector with e^{ix}, having Δx times of its value.



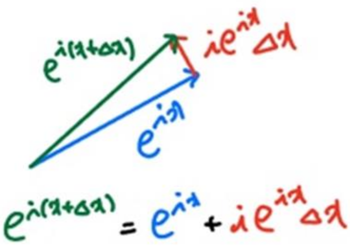
Applying the second rule of the complex plane operation, i$\Delta x e^{ix}$ is a point that $\Delta x e^{ix}$ is rotated 90 degrees on the complex plane.



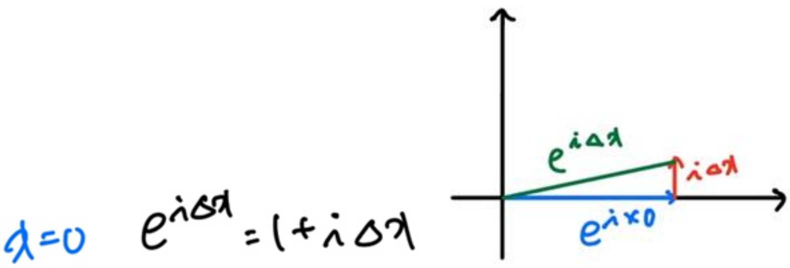


These two values can be added as the sum of the vectors on the complex plane.

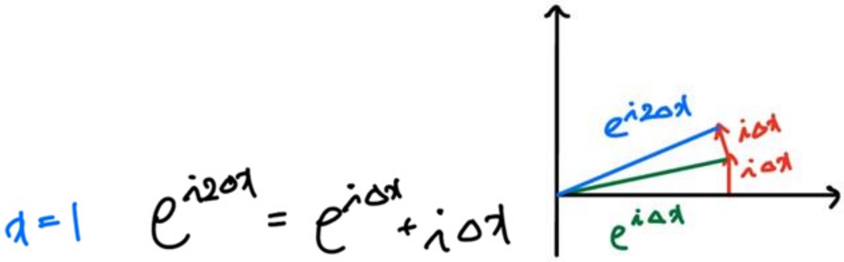
Since Δx is a very small value, the length of the arrow before and after multiplying i$\Delta x e^{ix}$ may be approximated.



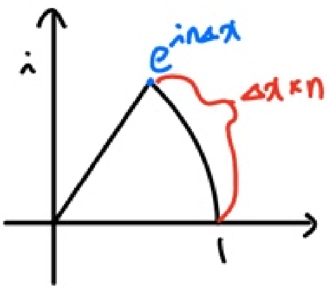
This rule holds for any point on the complex plane. Since the only point known is (1,0) when x=0, let's start from that point and apply the above rule to find the format of the graph. After substituting 0 for x, since e^0 is 1, we have the following equation. This is expressed on the complex plane as follows.



$i\Delta x$ has the same length as Δx on the complex plane. For an infinitely small Δx, the length of the arrow before and after this is added is equal. If Δx is substituted for x in the same way as we did earlier, it will be a value that is shifted by Δx in the direction perpendicular to the existing complex number.



When the complex plane is viewed in a polar coordinate system when x increases by Δx, e^{ix} moves by Δx in a direction perpendicular to the existing complex number (the direction in which theta increases).



In this way, the traces drawn by e^{ix} on the complex plane are circles with a radius of 1. After a glance at the points on the graph is taken, a circle on the complex plane is evident. Why is the trace on the graph a circle? The above relational expression is equivalent to the fact that the trace is a circle. One of the important features of a circle is that its radius and tangent are perpendicular at any point. When taking the points in the graph above, e^{ix} is the radial direction, iΔx is the tangential direction, and it is perpendicular to the direction of e^{ix}. Since a figure whose radius and tangent line are always perpendicular is a circle, the trace of e^{ix} on the complex plane is a circle.

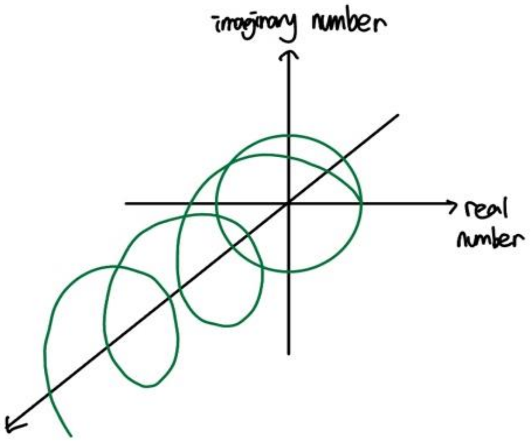
Moving on to the position of $e^{in\Delta x}$, which is the value when x increases n times by Δx to n Δx. The total length of the curve shifted up to this point is n Δx. This means the length of e^{ix} from x=0 is x. The length of the function represented by the variable x represents the angle to the real axis because it is part of a circle with a radius of 1. When x is π, $e^{i\pi}$ is located at the point where the real number part -1 and the imaginary number part 0 rotated by 180 degrees as the starting point. Therefore $e^{i\pi} + 1 = 0$, and the Euler's equation is derived. Through a complex sequence, the most beautiful equation is born.

Looking at the previous processes, you can feel the beauty of this formula. In the process of deriving Euler's equation, we were able to encounter a figure called a circle. A circle is a figure that represents geometry and has been considered the most sacred since the Middle Ages. Circles can be found on the earth, moon, eyeballs, and water droplets in which we live. A circle is sometimes represented by the value of π, which is expressed as the ratio of circumference to diameter. It is included as a constant in the Euler equation.

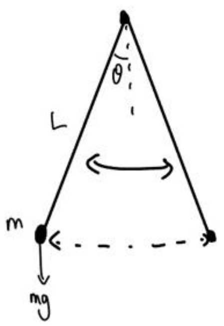
A circle has a periodicity. Analysis of the circle in which e^{ix} in space is projected onto the yz plane was mentioned above. In space, e^{ix} takes the form of a spiral. Once these spirals are projected onto the xy and xz planes, respectively, the form of a sine function appears. The Taylor expansion shows that e^{ix} can be represented by a linear combination of a sine and a cosine. Showing that respectively, e^{ix} is a periodic function.

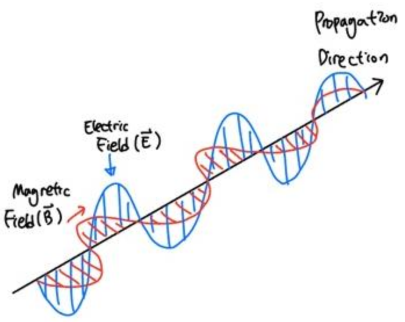
$$e^{ix} = \cos x + i \sin x$$

This is Euler's formula. If you substitute π in x, it's the Euler's equation.



Since e^{ix} is a periodic function, it is used to describe natural phenomena. For instance, the term $e^{i\omega t}$ can be used to express the solution of a single pendulum. In addition, the term $e^{i(k \cdot r - wt)}$ appears when describing light, an electromagnetic wave, and when mathematically expressing the propagation of electric and magnetic fields in time and space. Thus, in addition, when expressing the motion of microscopic objects as a wave function in quantum mechanics, the psi, wave function appears through the integral of e^{ikx}.





By solving the differential equation for the imaginary number i, a function with periodicity could be derived. The number 'i', which represents a complex number, contains the periodicity of natural phenomena.

After a close look towards the origin, the solution of the most basic differential equation e^x can be found. It is also found in natural phenomena that increase exponentially. The spread of infectious diseases, the spread of flames, and the proliferation of bacteria are all expressed in the form of exponential functions. The natural constant e is a very important and special constant in mathematically describing natural shapes.

As such, Euler's equation has natural constants 'e', imaginary number 'i', circumference π, and this simple equation in which 1 and 0 are connected only by four arithmetic operations. This is a beautiful representation of nature as a simplistic equation.