The Conundrum of the Casio Calculator
By Zack Ross, author of *What is a Kilogram?*

Casio Calculators. Anybody who has anything to do with maths has one. From basic pocket calculators to financial calculators to calculators with capabilities in calculus and matrices - Casio has them all. Casio is by far the one of the popular calculator brands around today however, what if I told you that there is an interesting fault in all of their calculators? You may scoff and suggest that such a popular brand could not possibly be flawed in one model - let alone all! You may laugh and tell me I am mistaken, and that the well-known Casio snake game is an intentional feature. Yet, I speak in facts. In this essay I will explain to you this mystery, theorise possible causes and crunch more numbers than I can ever say I wanted to. This is the conundrum of the Casio calculator…

Okay… moving swiftly on from the grandeur, I now hear you asking; what is this so-called mystery? Well, here you go, grab a calculator and follow along:

First, turn the calculator on - it won't work if you don't!

Second, type in to the calculator $\frac{11^6}{13}$.

Next, hit enter.

And finally behold…

A perfectly normal… wait what?!

The calculator has given an answer of $\frac{156158413\pi}{3600}$.

Now, all the non-geeks might be wondering - what's the big deal? It's actually pretty cool that such a simple entry gives a multiple of π!

However… this simply cannot be true. You see, π is what's known as an irrational number. But this means nothing, so I'll explain.

First we must understand the concept of number sets. A set in mathematics is simply a group of numbers, for example {3,5,-78,81} is a 4 number set. However, there are some much more important sets of numbers - important enough to have their own letter denotation - to be aware of:

The natural numbers (N): 1,2,3,4… (all positive integers)
The whole numbers (W): 0,1,2,3… (all non-negative integers)
The integers (Z): …,-3,-2,-1,0,1,2,3… (all digits which do not contain decimals)

The rational numbers (Q): e.g. $\frac{1}{2}$, 3, $\frac{7}{391}$, $\frac{-292}{3}$ (all numbers which can be expressed as the quotient of 2 integers, i.e. fractions).
The irrational numbers (P): e.g. π, $\sqrt{2}$, e, φ (any number which is not rational).

Now, given that π is irrational it cannot be expressed as a fraction, but this result would suggest that, after some rearranging, $\pi = \frac{3600x11^6}{156158413x13}$. So this can't be right.

But why! Why then, does the biggest calculator brand ever consistently design calculators which make this obscure mistake? There are many theories to this, one which is that the glitch is intentional so that the company can tell when somebody had stolen their software and, while this may be true, it wouldn't make for a very good maths essay! So we'll leave that theory aside and focus on some of the more intriguing ones.

Firstly; what if the calculator simply sees that the result is a very close approximation of π- Casios do only display (non-recurring) results to 10 decimal places, so this could make sense- and displays what it believes is the correct answer?
This makes even more sense when we evaluate what we previously derived to be "equal" to: $\frac{3600x11^6}{156158413x13}$.

This gives: 3.1415926535890389, whereas π is in actuality equal to
: 3.1415926535897932. That's accurate to 12 decimal places.

So that settles it! The calculator simply found a result which, within its computational limit, was a rational multiple of π. Done!

Right…?

No. Absolutely false.

Unfortunately for me, this very simple, basic, easy-to-explain solution is incorrect.
This is shown by evaluating $\frac{17^5}{11}$.

This fraction is extremely close to $\frac{366494029\pi}{8920}$ - so close in fact, that if we again derive π from this, we get a 13 decimal place accurate approximation! Even better than the previous!
However, Casio does not agree and instead gives us 129077.9091. Not a π in sight!
So what then? Is it just $\frac{11^6}{13}$?

No.

If we look at another example: $\frac{11^6}{17}$, which gives a result which approximates π to 12d.p. we see a decimal result. However, entering $\frac{11^6}{16.999999999995}$ gives us $\frac{119415257\pi}{3600}$. But why? Why does a result less accurate than $\frac{17^5}{11}$ give a multiple of π?

Well the answer seems to lie in the denominator.

Before we delve deeper we must first understand all of the circumstances under which the calculator displays an exact value of π. Obviously, if you simply press the π button then the calculator will display the π symbol. And then again if you type in, for example $\frac{2\pi}{2}$ it will give you π, or $cos^{-1}(-1)$ will give you π- assuming the calculator is in radians. Basically, any input which gives the exact value of π as the answer. Then obviously there are basic fractions of π; $sin^{-1}(1)$ or simply $\frac{\pi}{2}$ gives the answer of, well, $\frac{\pi}{2}$. This occurs up to and including $\frac{\pi}{10}$, but this is where it gets weird. $\frac{\pi}{11}$ does not give an exact value but instead 0.2855993321, however $\frac{\pi}{12}$ gives an exact value. This is weird. And it gets weirder when you consider $\frac{\pi}{14}$, another exact value, even though $\frac{\pi}{13}$ gives a decimal.

By crunching some numbers we can derive a list of all values of X, such that $\frac{\pi}{X}$ results in an exact fractional answer. All values of X less than or equal to 100 are: 1,2,3,4,5,6,7,8,9,10,12,14,15,16,18,20,21,24,25,28,30,35,36,40,42,45,48,50,56,60,63,70,72,75, 80,84,90,100.

So what's the pattern? There's seemingly no relationship between these numbers. Or is there? It is also worth noting that dividing any integer by one of these numbers and then multiplying by also gives a fraction of π, although this does depend on the size of said integer. This reveals that 3600 must be one of these numbers given our earlier results. And now we're cooking. Because, as it turns out, any factor of 3600 will be one of these numbers! You can check that yourself if you don't believe me. But that doesn't account for all of them. Last time I checked $\frac{3600}{7}$ wasn't an integer, nor was $\frac{3600}{28}$. So what's happening? Well, it turns out that 3600 isn't the biggest of these numbers. That title, as far as I can find, is 25200. All attempts to find a larger number have been fruitless, however there may end up being a larger integer.

So there we have it! Any fraction of π (with a numerator beneath a certain upper bound which seems to change depending on the denominator. Very frustrating.) in which the denominator is a factor of 25200, will display an exact result. Therefore, if a given input (such as $\frac{11^6}{13}$) is close enough to such a fraction, an exact fraction of π will be given. Unfortunately it has proven beyond my capabilities to determine the calculator's cut-off for what is and is not close enough to count.

However, that still leaves us with more questions than answers. Most importantly; why 25200?

Well, again, we don't know for certain. However, a leading theory amongst the only group of people I can find still investigating this (the comment section of the pinned comment on the YouTube video https://youtu.be/7LKy3lrkTRA?si=CLa0HXNXdPtvegC5- credit to Stand-up Maths for the essay idea.) is that it is related to angles.

For this we must understand the three angle units with which the calculator can work: degrees, radians and gradians. We all know about degrees, but to recap; there are 360 of them in a full rotation. Done.

Radians are a topic which most people only ever see in school and then happily erase from their memory the day after the exam- although if you're reading this that's probably not you, but I'll go over them anyway.

One radian is defined as the angle for which the arc length of the sector it governs is equal to the radii of the sector. So basically, if you take a one radian sector of a circle with radius r, the length of arc of that sector will be r. Or, even more simply, 1 radian equals 57.296 degrees and 360 degrees is equal to 2π radians. There's π again, see where this is going?

The gradian, however, is a far more obscure one which I must admit I had to research the definition of before making this essay. Thankfully however, it is very easy to understand; a right angle (90 degrees, $\frac{\pi}{2}$ radians) is equal to 100 gradians.

Now we must look at the conversion factors (the ratios by which the different units can be converted. Be don't really care about deg to grad, however the other two are useful.
1 deg = $\frac{\pi}{180}$ rad and 1 grad is equal to $\frac{\pi}{200}$ rad.

But how does this help? This becomes clear once we understand the concept of highly composite numbers.

(I know, here we go again with the subtopics.)

A highly composite number is defined as a natural number which has more divisors than any number smaller than it. For example; 12 is a highly composite number as it has 6 divisors -1,2,3,4,6,12-, more than any number 1-11. And, surprise surprise who would've thought - 25200 is also a highly composite number.

But, so what? Why does this help?

Well, it would be very beneficial for the calculator to be able to express $\frac{\pi}{X}$ as a fraction for as many X's as possible, while minimising the code required to code this. So instead of the calculator simply having a line of code telling it to express $\frac{\pi}{X}$ as a fraction of π for all integers 1,2,3… on and on and on, it simply has one telling it to express $\frac{\pi}{X}$ as a fraction for all factors of

25200! Well, we think. This is another instance where we likely will never know for certain unless we get a job at Casio, but we can still theorise.

However, this still doesn't relate to the deg-rad-grad stuff from earlier. But it does! Because, as luck would have it, 25200 is the smallest h.c.n. which is divisible by both 180 and 200! It would obviously be necessary for the calculator to be able to convert between the three angle units it operates in so we could onto something. It also helps that 25200 is also divisible by all numbers 1-10, values for which it would obviously be of benefit to receive exact values. But why not bigger? 50400 is also highly composite, divisible by 1-10, 180 and 200 and even has 108 divisors compared to 25200's 90! Well - can you guess what I'm going to say? - we just don't know. If we are correct that 25200 is significant, then it is likely because 25200 is some sort of perfect equilibrium of having enough factors, having the necessary factors and not requiring too much data to code. So there we have it, right?

Well, almost. I still haven't spoken about why the calculator falsely approximates π! We've discussed the criteria for which a fraction of π will be displayed, but haven't spoken about the fact that the calculator is still displaying these fractions where it shouldn't!

Again, we don't know (bit of a running theme here, isn't it?) however there are some interesting features of the π constant which we should consider. Firstly, as I mentioned earlier, the calculator displays π as 3.141592654 (press the s⇔d button to show this instead of the symbol). But if we subtract 3.141592654 from this answer, what do we get? Zero, obviously. Not!

Instead we are given $-4.102x10^{-10}$. Okay… so the calculator stores π to more decimal places than it displays, odd, but nothing groundbreaking. However we can easily determine this number to be 12 by doing the following sums (on the calculator, of course, don't go reaching for a pen):

π-3.14159265358 (11d.p. of π) =$9.8x10^{-12}$.
π-3.141592653589 (12d.p. of π) =0.
So the Casio stores π to 12 d.p. Big deal!

Except that it is! You see the calculator doesn't start rounding small numbers to zero until they reach the order of magnitude of 10^{-100}. This implies that the calculator doesn't store π further than 12 d.p. and just round the tiny remainder to zero, but instead actually only has 12 decimal places of π in its code! This isn't a dig at Casio or anything - the more decimal places you store the more complex the code becomes (I think, I've never done coding. At least not successfully.). So this is where those errors come from. When we're dealing with numbers to the scale of 11^6, even small discrepancies at the thirteenth decimal place can cause major errors in our approximations.

So there we have it, finally something concrete - the calculator only stores π to a, relatively, small number of decimal places. This allows rounding errors to trick the calculator into displaying fractional multiples of π for certain equations, if and only if the denominator of that fraction is a factor of 25200.

Phew. That was a lot of numbers! Still awake? I won't be for long, I need sleep. Or some pie…

Thanks for reading!