A very interesting topic in mathematics is the study of series and their applications and appearances in nature. As someone who plays a musical instrument, the example of harmonic series is of particular interest, as it manifests itself in the way a string/column of air vibrates to produce a musical tone. All it is, is the sum of 1/n from 1 up to some point, and a string/column of air vibrates over the same series, based on its length, with each frequency at a different strength leading to what musicians call overtones: these govern the unique timbre of an instrument! To write that mathematically, it is the following:

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Or as a sum:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n}$$

Rather simple right? Well music aside, what else could we say about it? What interesting properties does it have? Well a common interest amongst mathematicians when it comes to a series is to ask whether it converges or diverges when adding up an infinite terms. To put that simply, does adding up terms from n=1 to infinity eventually get closer and closer to a certain value, never surpassing it, or does it increase without bounds, basically reaching infinity itself? Intuitively, you might say that since you add smaller and smaller terms each time, soon so unimaginally tiny it barely makes a difference, it must surely converge to some point, but that is surprisingly not the case, as proven by a Frenchman Nicole Oresme in 1350.

Lets set out the original sum again:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \dots$$

Now lets change a few terms:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \dots$$

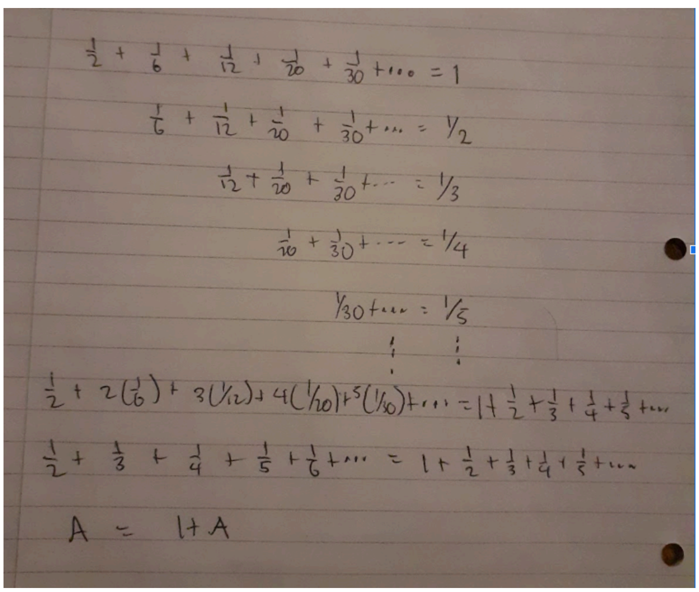
Now which series is smaller, the original or the new altered one? Clearly the latter, as terms either stay the same or are reduced so the denominator is the next power of 2: 2, 4, 8 etc. But if you group them as so:

$$1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{1/2} + \underbrace{\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}}_{1/2} + \dots$$

You actually end up with 1 + ½ an infinite amount of times if you go to infinity for the latter series, as you'll have 1/16 8 times, 1/32 16 times etc. Generalising for a specific term for this new one, we get $S_{2^n} = 1 + n/2$. Inserting infinity as n clearly indicates a divergence. But as previously shown, our original harmonic series is actually greater than our new one which diverges. Since the series is greater than one which diverges to infinity, we can conclude that the harmonic series does indeed diverge to infinity!

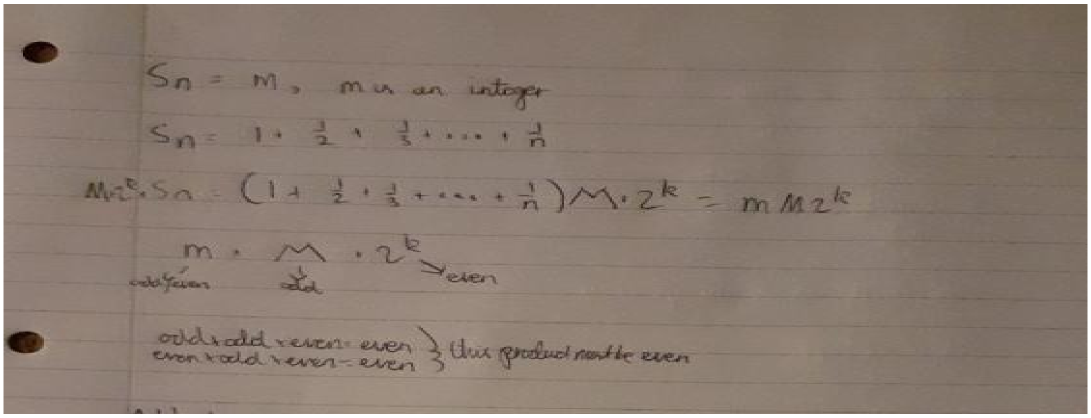
There is actually more than one method to prove this, though Oresme's method is the most well-known. Lets take another example, first used by Jacob Bernoulli and credited to his brother: take as a prerequisite that ½ + ⅙ + 1/12 + 1/20 + … = 1.
Now, let there be some series A such that ½ + ⅓ + ¼ + ⅕ + … = A.

Follow this proof below by subtracting the initial term in each following series to write a new one and then adding all the equations together

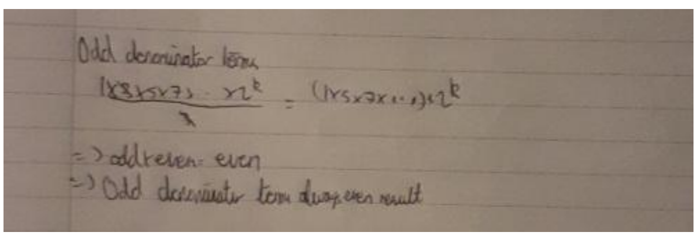


The final idea that ½ + ⅓ +¼ + … = 1+ itself cannot be true for any finite value, indicating 1+A (the harmonic series) is divergent!

The next interesting feature is the question of at what point you can add up the terms to that point and have it be an integer. Naturally for 1, it is obviously an integer, but for any successive term will you reach a whole number? Well because you pass across every single positive integer up to infinity you might say yes you can, but let us then try to prove that S_n (the sum of harmonic numbers up to 1/n) can be an integer when n>1. I shall explain at the end why this doesn't exactly work for n=1. We must follow the proof below, by assuming that the sum up to that number =m, where the only special thing about m is being an integer. We want to multiply both sides of the equality by something to help us, so we will times both sides by M (all odd numbers less than or equal to n), and 2^k, which is the largest power of 2 less than or equal to n). The reason will become apparent shortly. As shown in the below picture, any integer m will multiply with M and 2^k to give an even result:

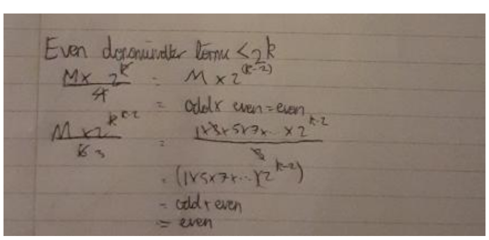


As for the actual sum multiplied by M and 2^k, how are odd denominators affected(⅓, ⅕ etc)?

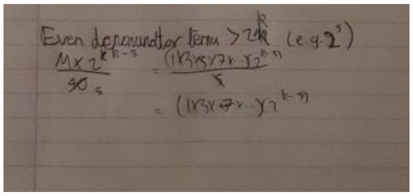


Every denominator will be in M so they cancel, leaving still an odd term. This multiplies with the even term to give another even one as shown. Take the example of the denominator of 3 given a larger value of n

So any odd denominator term is even. But what about even denominators? Well for denominators smaller than 2^k, they cancel out through 2^k, leaving another even number multiplied by an odd. Take the example of 4 given a larger value of n e.g. 16, and 6.

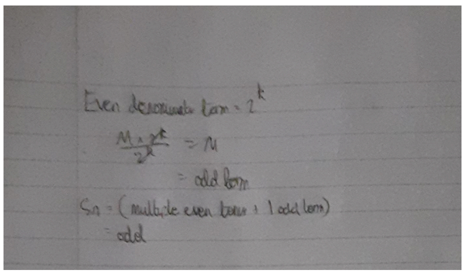


Thats always an even term as well! Now what about even denominators greater than 2^k? Let us see how a term with a denominator like 40 evaluates, given 40 is greater than 2^k. An example of the denominator 2^k here could be 2^5 (32)



This has even * odd which is even! So all terms so far (except 1/(2^k)) are even when multiplied by M and 2^k.
The sums possibility to be integer or not rests solely with this final term.

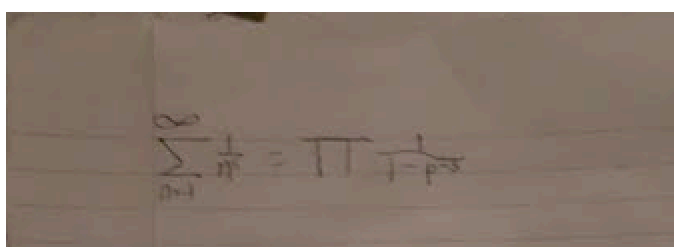
By following the method , we find out the term is equal to M as shown below



So this single term will be odd! Now since M*2^k *S_n has all even $_n$terms except one, its overall sum will be odd. But we found out previously that all integers will multiply with M and 2^k to give an even result. So there must be a contradiction in our argument somewhere. The only possible explanation is that our previous statement S_n and m are equal is not true. But the only special thing about m was being an integer. Therefore S_n is not an integer! Now as I said before, for the case n=1, it is slightly different. The original idea of m*M*2^k for n=1 would mean 2^k = 2^0 = 1. But then M*2^k is odd and if m is odd or even, the RHS could be odd or even too. So having the LHS being equal to an odd number with 1*M*2^k does not cause any problems.

Now we have talked a lot about the interesting theoretical properties of the harmonic series, we should try to find some applications in the real world or to other parts of mathematics. One application is through something rather dissimilar: the infinitude of the prime numbers as proven by Leonhard Euler in the 18th century

For this proof, we shall want to prove this equality shown: proving the general case of s will allow us to assume the equality for s=1, hence getting the harmonic series on the LHS



$$\sum_{n=1}^{\infty}\frac{1}{n^s} = 1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\dots$$

First take the left hand side and expand it.

$$\frac{1}{2^s}\sum_{n=1}^{\infty}\frac{1}{n^s} = \frac{1}{2^s}+\frac{1}{4^s}+\frac{1}{6^s}+\frac{1}{8^s}+\dots$$

Multiply both sides by 1/(2^s)

$$\left(1-\frac{1}{2^s}\right)\sum_{n=1}^{\infty}\frac{1}{n^s} = 1+\frac{1}{3^s}+\frac{1}{5^s}+\frac{1}{7^s}+\dots$$

Subtract the second equation from the first

$$\frac{1}{3^s}\left(1-\frac{1}{2^s}\right)\sum_{n=1}^{\infty}\frac{1}{n^s} = \frac{1}{3^s}+\frac{1}{9^s}+\frac{1}{15^s}+\frac{1}{21^s}+\dots$$

$$\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{2^s}\right)\sum_{n=1}^{\infty}\frac{1}{n^s} = 1+\frac{1}{5^s}+\frac{1}{7^s}+\frac{1}{11^s}$$

Doing the same for 1/(3^s) shows this pattern continues for all prime numbers, as a similarity to the sieve of erasothenes, where taking out all multiples of primes will eventually take out all numbers, as any non-prime number can be expressed as the product of primes. Since the terms (1-1/p^-s) are multiplied, we can represent this product as so:

$$\prod\left(1-1/p^s\right)\sum_{n=1}^{\infty}\frac{1}{n^s} = 1$$

Dividing both sides by the product you get the final result we are trying to prove. Taking s=1, we get the following

$$\sum_{n=1}^{\infty}\frac{1}{n} = \prod\frac{1}{1-1/p}$$

Now we want to manipulate the right hand sign to better prove the infinitude of primes. However we cannot change whether the final result diverges or not, so we are going to take the right hand side and multiply it by the natural log. This is because log(abc..) = log a + log b+ log c. Why this is acceptable is because because the logarithm of a convergent series is always finite, and for a divergent it is still infinite, so it won't affect our final result of divergence, and it simplifies our calculations as you shall find out:

$$\ln\left(\prod\frac{1}{1-1/p}\right) = \sum\left(\ln\left(\frac{1}{1-1/p}\right)\right)$$
$$= -\sum\left(\ln\left(1-1/p\right)\right)$$

Now unfortunately I must take a prerequisite here as its proof requires knowledge of the Taylor series, and to try to explain this in an understandable manner would require much explanation and I would prefer to stick close to the topic in question. The assumption is that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

Taking x= -1/p, we get the following results as shown below.

$$-\sum \ln(1 - \underbrace{p^{-1}}_{x}) = -p^{-1} - \frac{(-p^{-1})^2}{2} + \frac{(-p^{-1})^3}{3} - \frac{(-p^{-1})^4}{4}$$

$$= -\frac{1}{p} - \frac{1}{2p^2} - \frac{1}{3p^3}$$

$$-\sum \ln(1-p^{-1}) = -\left(\frac{1}{p} + \frac{1}{2p^2} + \frac{1}{3p^3} + \dots\right)$$

But the minuses on either side of the equality cancel and the terms aside 1/p become so small they converge, hence allowing us to rewrite the equation so:

$$\sum \ln(1-p^{-1}) = \sum \frac{1}{p} + K$$

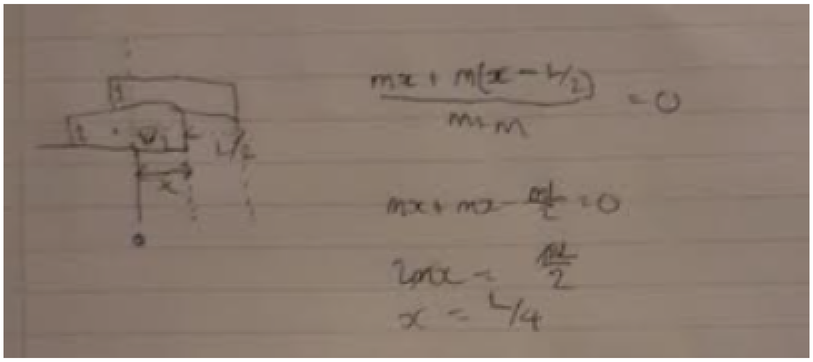
$$\sum_{n=1} \frac{1}{n} = \sum \frac{1}{p} + K$$

As the harmonic diverges, and the right hand side does the same as we haven't manipulated the side to change that property, and K is convergent, in order to have a divergent RHS, the sum of reciprocal primes must be infinite. But to be infinite you need an infinite number of terms. So there must be an infinite 1/p, and concurrently an infinite number of primes!

A further application is through the block stacking problem. When you stack uniform blocks on the edge of a table, how much of an overhang is possible?

Well we need to get into some mechanics. Lets start with the simplest number of blocks: 1. Assuming the block has mass distributed evenly throughout, the centre of mass will be at the geometric centre of it. For the furthest overhang, this must be exactly on the edge of the table, so if the length of the block is L, the overhang is ½ L.

For two blocks, we need two things: block 1 to balance as far as possible over block 2, and block 1 and 2 combined to balance just on the edge. To have block 1 just over block 2 simply resembles the original block over edge- a distance of L/2. But for block 1 and 2, we need the midpoint of the total overlap of the blocks if it is directly over the edge:(L/2 + L)/2 = 3L/4, or ½ L + ¼ L



But what about three blocks? Well we do a similar thing: block 1 overhang over 2 is L/2, the combined of 1 and 2 is L/4. But for three blocks, how does it work? Relatively similar. Following the same pattern, the overlap of three blocks over the edge would be 11/12 L, or ½ L + ¼ L + ⅙ L. Continuing this pattern for all numbers n will get ½ L + ¼ L + ⅙ L + ⅛ L + … or L/2(1 + ½ + ⅓ + ¼ + …+ 1/n). How far can this stretch? Well the harmonic series that has manifested itself here goes to infinity, so multiplying L/2 by infinity gets infinity. Therefore, provided you have enough blocks, you can have any overhang distance, and it never tails off! You could stack them from one corner of the room to another, London to New York or even to the edge of the universe- theoretically speaking of course. I highly doubt that would be practically possible.

To conclude, I have given here a rather brief and simple introduction to the wonders of the harmonic series. I have given some theoretical aspects I find fascinating, and given 2 examples, that have physical applications. These examples illustrate how the world of mathematics finds expression in the natural world.