AN INTRODUCTION TO VECTORS

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Introduction

Vectors. In mathematics, we have been taught how to find the magnitude of a vector or perform other operations on vectors, but what actually is a vector? A formal definition for a ***vector*** would be '*a quantity having direction as well as magnitude*' which you may be familiar with from school.

However, vectors have uses outside of school – in engineering, vectors help with designing structures and the analysis of forces, and in computer graphics, vectors can be used to create realistic animations, control movement and simulate lighting. Even in artificial intelligence, vectors are used in the handling of data in the neural networks of machine learning algorithms. ***Vectors*** can be thought of as ***arrows*** in space and are powerful 'tools' very useful in coordinate geometry.

When creating concepts, one approach would be to construct a set of objects and rules to manipulate these objects. This is known as ***algebra***. ***Linear Algebra*** is the study of ***vectors*** and certain rules used to manipulate vectors and is what is going to be covered in this essay.

Vector addition & Scalar multiplication

The vectors that most of us know from school are called 'geometric vectors' and can be denoted with an underlined letter e.g., $\underline{x}$ and $\underline{y}$. They are usually represented either through ***column vectors*** containing ***elements*** or through ***unit vector*** notation where the number of elements or coordinates determine the ***dimension*** of that vector.

$$\underline{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\underline{x} = i + 2j + 3k$$

Vectors can undergo ***addition*** such that $\underline{x} + \underline{y} = \underline{z}$ is also another geometric vector, where each element of $\underline{z}$ is the sum of each corresponding element from $\underline{x}$ and $\underline{y}$ (however both $\underline{x}$ and $\underline{y}$ must be of the same dimension). Moreover, ***multiplication*** by a ***scalar*** $\lambda\underline{x}$, $\lambda \in \mathbf{R}$, is also considered a geometric vector, but results in each element of the original vector being scaled by λ. These two operations (***vector addition*** and ***scalar multiplication***) are crucial when describing linear combinations.

$$\underline{y} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix},$$

$$\underline{z} = \underline{x} + \underline{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}.$$

$$3\underline{y} = \begin{pmatrix} 6 \\ 9 \\ 12 \end{pmatrix}$$

Magnitude & Direction

Two important properties of vectors that we mentioned earlier were its ***magnitude*** and ***direction***. The ***magnitude*** of a vector defines its ‘size’ or ‘length’ and is a scalar value independent of the vector’s direction. This can be calculated by applying the ***Pythagorean theorem*** on the elements of a given vector:

$$|\underline{a}| = \sqrt{a_1^2 + a_2^2}$$

(The magnitude of a vector is also denoted by a ***pair*** of ***lines*** either side of the given vector).

This also applies in three dimensions:

$$|\underline{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

So we have general formula for n dimensions:

$$|\underline{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2}$$

Using the magnitude of a vector we can find its ***unit vector***. A ***unit vector*** is a vector that has a magnitude of **1**, examples of unit vectors include: i, j and k (where each of these unit vectors represent the unit vectors in the x, y and z direction respectively).

To obtain a unit vector in the same direction of a given vector $\underline{v}$, you divide this vector by its magnitude:

$$\hat{v} = \frac{\underline{v}}{|\underline{v}|}$$

(The unit vector is usually denoted with a little ***hat*** on top of the originally defined vector).

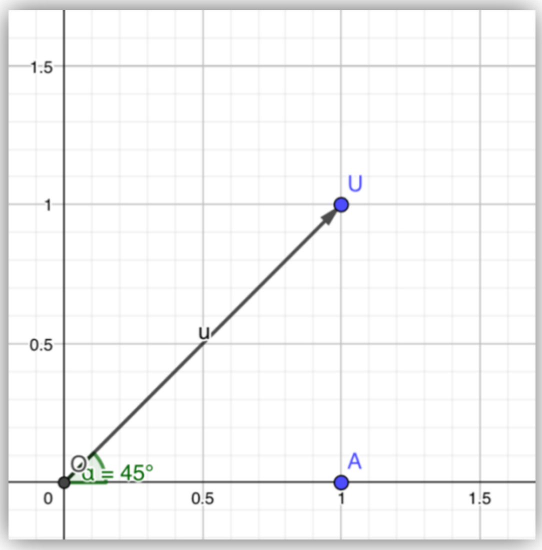
Hence finding the unit vector of $\underline{x}$:

$$|\underline{x}| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$

$$\hat{x} = \frac{\underline{x}}{|\underline{x}|} = \frac{1}{\sqrt{14}}\begin{pmatrix}1\\2\\3\end{pmatrix}$$

Furthermore, the ***direction*** of a vector can be described by its angle with another vector or a reference axis. In two dimensions, this is often measured as the (anticlockwise) angle between the vector and the positive x-axis, but in higher dimensions angles between planes can describe the direction of a vector.

Thus, we can use ***trigonometry*** to find the angle a 2D vector makes with the positive x-axis:



$$\underline{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\tan \alpha = \frac{u_y}{u_x} = \frac{1}{1} = 1$$

$$\alpha = \arctan(1) = 45^\circ \quad \text{(or } \frac{\pi}{4} \text{ radians)}$$

The dot product

The ***dot product***, also known as the ***scalar product***, is an operation that takes two vectors and returns a single number by multiplying the corresponding elements in each vector and taking their sum.

It is defined as:

$$\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n \, .$$

Where:

$$\underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

So we can find the dot product of $\underline{x}$ and $\underline{y}$:

$$\underline{x} \cdot \underline{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 = 2 + 6 + 12 = 20$$

The ***dot product*** can also be expressed geometrically:

$$\underline{a} \cdot \underline{b} = |\underline{a}||\underline{b}| \cos \theta$$

(Where $\underline{a}$ and $\underline{b}$ are vectors and $|\underline{a}|$ is the ***magnitude*** of $\underline{a}$, $|\underline{b}|$ is the ***magnitude*** of $\underline{b}$ and θ is the ***angle*** between the vectors).

Therefore, the dot product can be used to find the ***angle*** between two vectors. If the dot product is ***zero***, it indicates that the vectors are ***perpendicular*** to each other. This property of the dot product is really useful in computer graphics – rendering brightness and colours uses the dot product to calculate the intensity of light on a surface.

However, if the two vectors are parallel, the angle θ between them is 0 and therefore cosθ = 1 and the dot product becomes maximised. Conversely, if the two vectors are in opposite directions the angle between them is 180° which means cosθ = -1 and the dot product becomes minimised.

The dot product also has important properties that may be used when dealing with vectors:

1. Commutativity

$$\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$$

The dot product is ***commutative*** which means the order of the vectors does not matter – the same result is returned regardless.

2. Distributivity

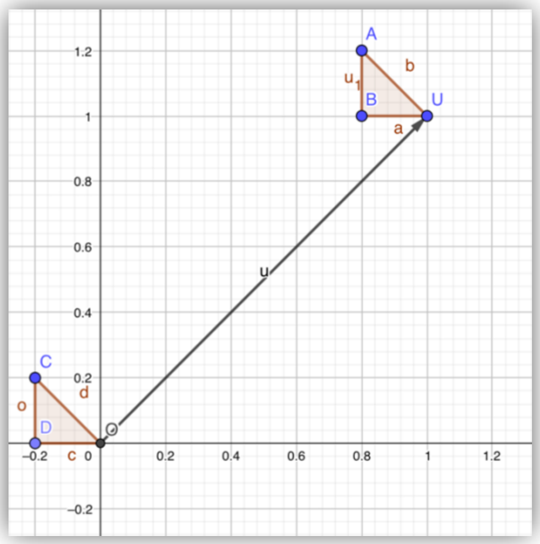
$$\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$$

The dot product is ***distributive*** which means that the dot product of a vector with the sum of two vectors is equal to the sum of the dot product of the first vector with the other two vectors.

Translations

Translations! From as young as when you were in primary school you may have dealt with moving shapes up, down, left and right on a coordinate grid. But how would we describe translations? Well, vectors provide a concise and efficient way of describing translations since they are able to represent both the magnitude and direction of a movement. Think of a translation as a transformation that shifts every point on an object in the same direction and distance and the ***translation vector*** as a way of showing what movements are required to do the translation.

Let's visualise this:



Here is a right-angled triangle being translated by the vector $\underline{u}$.

If we treat the coordinates of the right-angled triangles as ***position vectors*** we would be able to perform ***vector addition*** with the the coordinates as position vectors and the ***translation vector*** $\underline{u}$.

Let's try this with the coordinates D(-0.2,0) and our translation vector $\underline{u}$:

$$\underline{d} = \begin{pmatrix} -0.2 \\ 0 \end{pmatrix}, \quad \underline{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\underline{b} = \underline{d} + \underline{u}$$

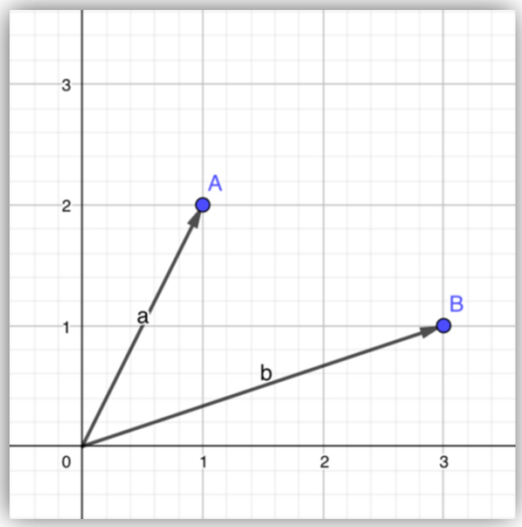
Calculating for b by translating D(-0.2, 0) by $\underline{u}$:

$$\underline{b} = \begin{pmatrix} -0.2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -0.2 + 1 \\ 0 + 1 \end{pmatrix} = \begin{pmatrix} 0.8 \\ 1 \end{pmatrix}$$

Hence the coordinates are B(0.8,1).

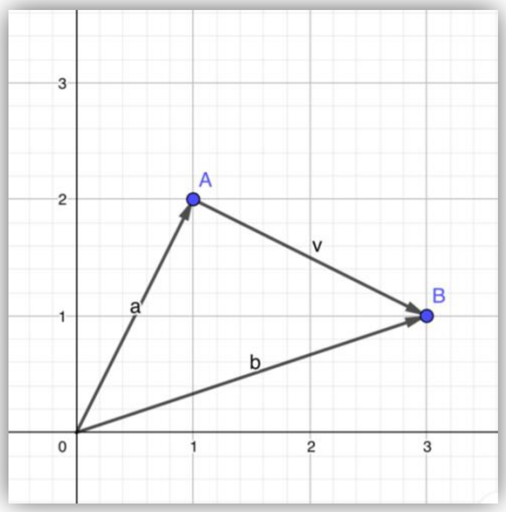
Finally, let's try to understand how vectors work in more detail. Suppose we have two vectors $\underline{a}$ and $\underline{b}$ such that:

$$\underline{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \underline{b} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$



How would we find the vector A to B by only using the given vectors $\underline{a}$ and $\underline{b}$?

We would have to go from A to O then from O to B, which in terms of our given vectors would be -$\underline{a}$ and $\underline{b}$ respectively. Thus, the sum of these two vectors becomes the ***resultant vector*** A to B:



Therefore, we can calculate this by doing:

$$\overrightarrow{AB} = \underline{b} - \underline{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

Now we have our resultant vector that takes from A to B and it becomes clear that when dealing with resultant vectors or vector addition you should think of the pathways given to create that same resultant vector.

Conclusion

Overall, my essay, ‘an introduction to vectors’ conveys the concepts and ideas that I have found interesting whilst learning about vectors both inside and outside school. I also hope that this essay has not only taught you a new way to look at vectors but also made you more curious on how vectors may be used outside of education.

Thank you for reading!