

# The Hat, The Proof, and Aperiodic Tilings' Beauty

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## Introduction

What is beautiful about maths? Is it a proof that seems complicated at first and yet unfurls into a simple elegance? Or is it the way so much of the world can be understood by the numbers we assign? One example of the beauty of maths is its applications to art. Aperiodic tessellations are a great example of using math to create art. A tessellation is a way to cover the plane without any gaps or overlaps. An aperiodic tessellation is a tessellation that has no translation symmetry, meaning that any way you shift the tiling it will never line up with itself again. We may be more familiar with simple grids using squares or triangles, which are satisfying in their own way, but aperiodic tessellations have a unique geometry that is intriguing in its own right.

## History

It is not easy to find sets of tiles that cover the plane aperiodically. In fact, it was only in the 20th century that the first set of tiles fulfilling this role was discovered. In 1966, Berger discovered the first aperiodic tessellation. It required over 20,426 unique tiles to achieve. The number of tiles needed was slowly whittled down over a decade. Once Penrose found his two tile set to tile the plane aperiodically, as seen in figure one, progress plateaued.

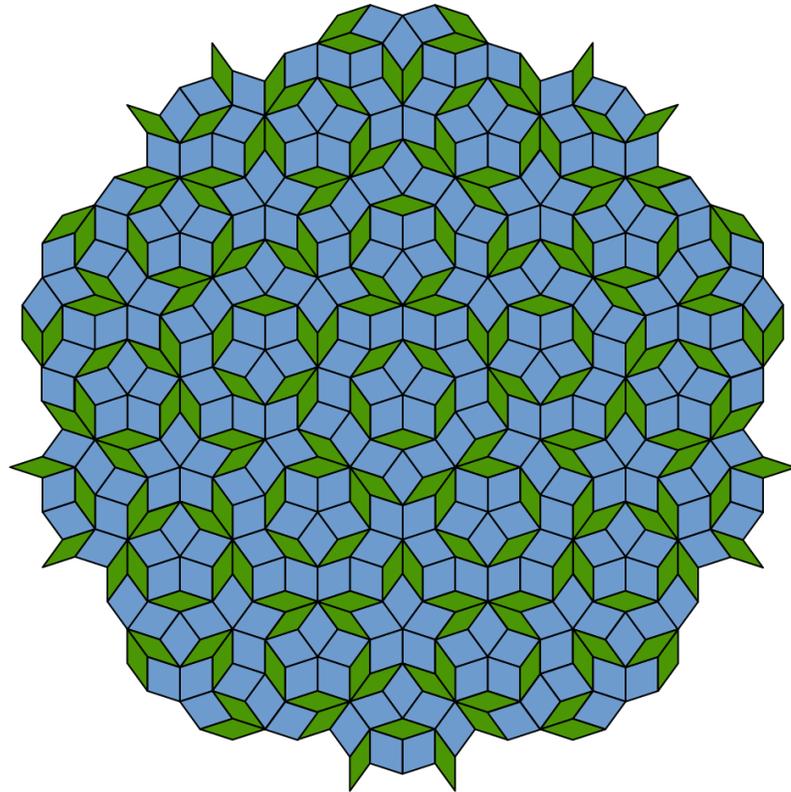


Figure 1: Penrose tiling, as far as the set of tiles had been reduced until the discovery of the hat

Credit: Wikimedia Commons, by Inductiveload

It took around 50 years to finally find an aperiodic monotile, where only one tile is used in the tessellation. In 2023, the hat tile was discovered, showing that the plane could be tiled aperiodically by a single tile. While this was a major breakthrough, there was still a caveat to the solution - the hat tiling required both the tile and its reflection to tile the plane infinitely. Is a shape and its reflection the same shape? While this question doesn't have a clear answer, the mathematical community did not have long to deliberate on whether the hat tile solved this long lasting puzzle. Only a few months later, the same team found what they called the spectre. This

new tile does not need its reflection to tile the plane aperiodically. Really, it is an altered version of the hat. If you adjust the side lengths of the hat to be equal, while keeping the angles the same, you get a tile called Tile(1,1).

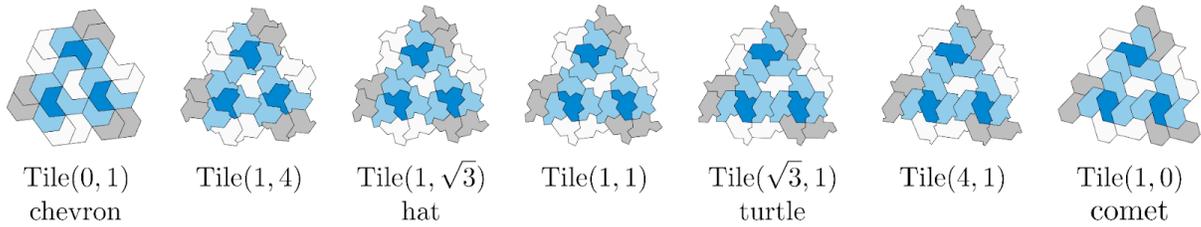


Figure 2: Hat tile continuum, showing what happens when side lengths are adjusted.

Tile(1,1) sits in the middle, where sides have equal length

Credit: An Aperiodic Monotile, David Smith et al.

This tile tiles the plane both periodically and aperiodically, which means it cannot be an aperiodic monotile. Quite cleverly though, the team decided to curve the edges of Tile(1,1), ending up with the spectres. Once the edges are curved, the reflection cannot tile with the original because there would be a gap, and yet the tile fits with itself. Finally, there was an aperiodic monotile that did not require reflection.

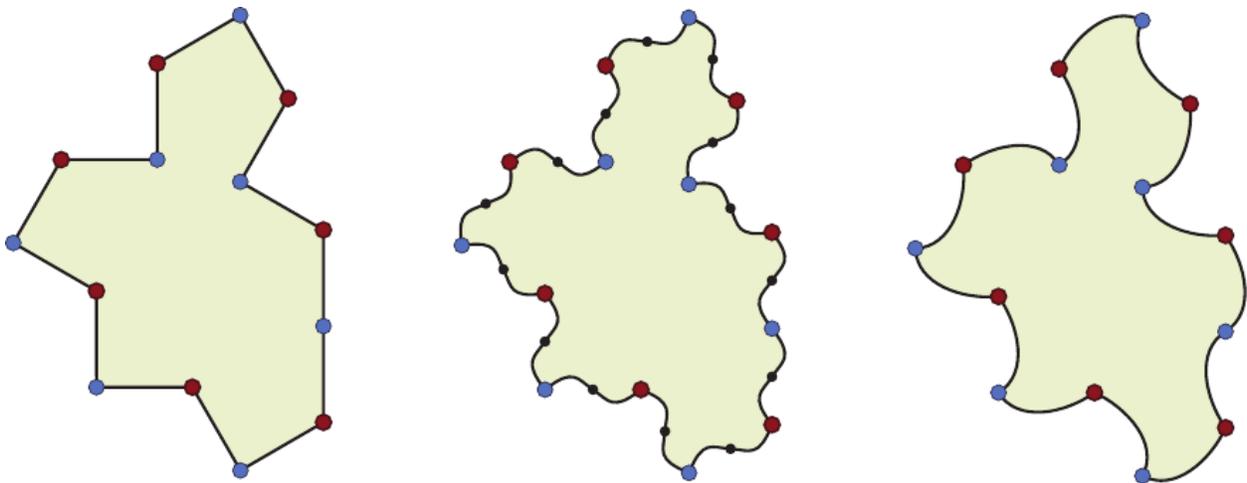


Figure 3: Showing how the spectre is made by curving the edges of Tile(1,1)

Credit: A Chiral Aperiodic Monotile, David Smith et al.

## The Proofs

In the original research paper by David Smith et al., two completely different proofs are used to show that the hat tiles the plane aperiodically. The first was based on a proof that has been used widely since Berger's first aperiodic set of tiles to prove various aperiodic tilings. This first proof involves proving that tiling has a hierarchical structure - a structure that repeats as you zoom further and further out, specifically in such a way that one smaller tile will belong to only one metatyle, the name for the next level up in the hierarchy. Each metatyle then only fits into one supertyle, and this pattern repeats for larger and larger groupings of tiles.

We can compare it to normal, periodic tiles to understand this better. If you have a square grid, each square belongs to a larger 2 by 2 square. Why is this not aperiodic? It lies in the fact that the square lies within multiple larger 2 by 2 squares. You can take your starter square as any corner of the larger square. In an aperiodic, hierarchical structure, the smaller unit of tile belongs to one and only one metatyle. This is the core as to why there isn't any translational symmetry.

To let this idea sink in, we can think of the contradiction that makes this work. If the tiling were to have translational symmetry, there would be a vector such that moving the tiling along that vector would line up with the tiling again. We can pretend as though we are to find the magnitude of said vector for our hat tiling. In order for the translation to line up, the vector has to be greater than the largest unique group of tiles. But, as described before, this tiling has infinitely larger unique metatyles. Thus the vector would have to be impossibly big and thus there is no

vector that would result in translational symmetry.

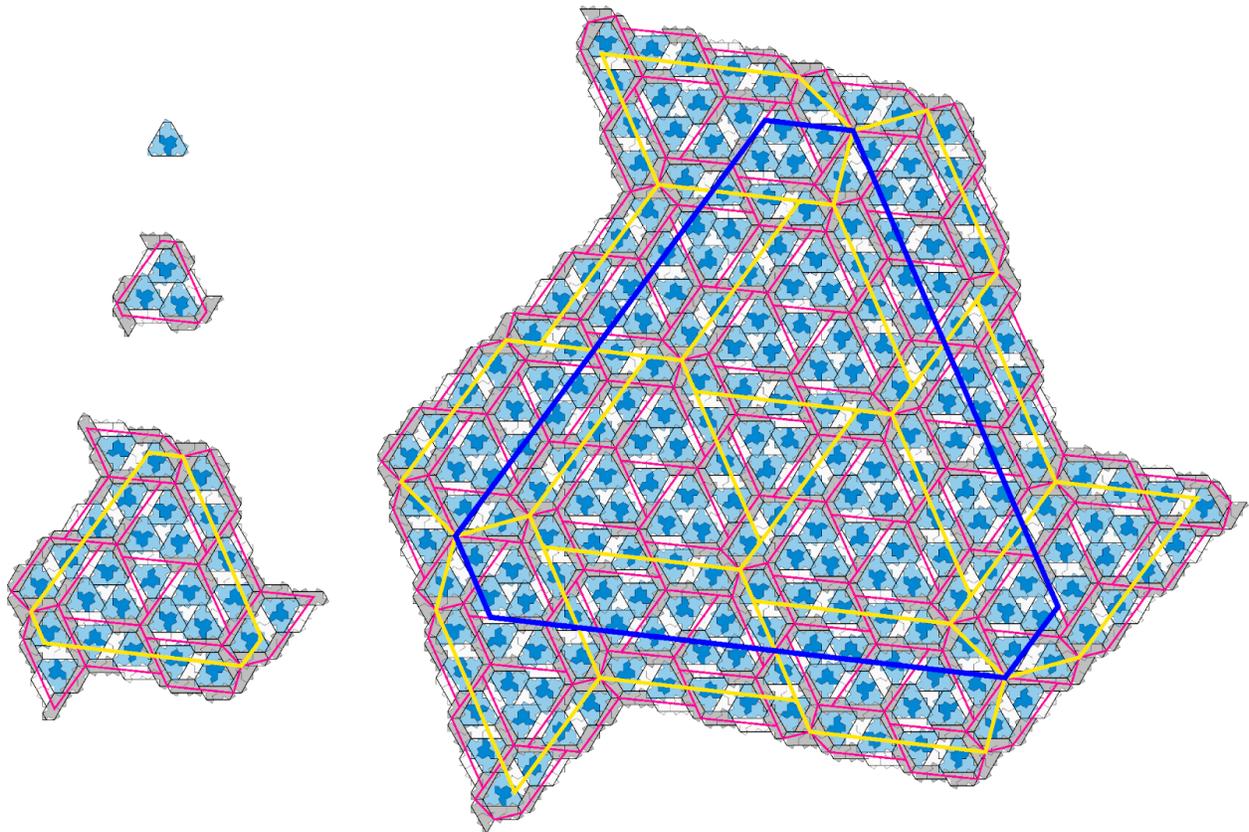


Figure 4: Shows the hierarchical repetition of the hat tessellation

Credit: An Aperiodic Monotile, David Smith et al.

The second proof takes a completely novel approach to proving aperiodicity. It requires a bit more information. The hat tile is a shape with 13 sides of lengths 1,  $\sqrt{3}$ , and 2. With a little trick, the side of length 2 can be thought of as 2 lengths of side length 1. If the side length of 1 is adjusted to zero keeping all angles the same, you get a shape called the chevron. Similarly, if you adjust the lengths of the  $\sqrt{3}$  to be zero, keeping angles constant you get a shape called the comet, as shown in Figure 2. Both the comet and the chevron tile periodically. However, the comet and the chevron cannot tile together without creating gaps or overlaps. As the hat tile is on a

continuum between these two periodic shapes that can't tile together, it has a tiling that is a sort of middle ground. It can tile the plane, but only aperiodically.

## Real Life Applications

The primary reason for this paper was to simply show that such a monotile that tiles the plane aperiodically can exist. Yet, these aperiodic patterns appear in the real world too. In materials science, aperiodic tilings are connected to the structure of quasicrystals – materials that have no repeating unit. Quasicrystals were first discovered in the 1980s. Some understanding of the properties of quasicrystals came from Penrose's tiling. Still, finding an aperiodic monotile is an important advancement to understand more of what type of materials could be made and what their properties would be. They're also just... beautiful. Penrose tilings have already shown up in public art and architecture, like floor patterns and murals. There's no reason that hat and spectre tilings couldn't do the same. The fact that these tilings never repeat gives them a strange kind of visual richness. At first, some structure seems to appear from the image, some pattern, but upon closer inspection the aperiodicity means the pattern never quite lines up, which is beautiful itself.

## Conclusion

The discovery of the hat and the spectre tiles is an example of mathematics at its best: a curious question leads to an unexpected answer, and suddenly we see things in a whole new way. For decades, mathematicians wondered whether a single tile could force aperiodic tiling. The hat showed that it could, even if it needed a mirror image to do so. Then the spectre went one step further – showing that even without reflection, a single shape could tile the plane aperiodically. Now there are further questions revealed by the discovery. Is there a systematic way to discover

aperiodic monotile? Is there an equivalent in 3D? Indeed, the continuous search for answers and inevitable discovery of further questions is what makes maths beautiful.

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