Are Imaginary numbers the only ‘non real’ numbers?

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1 Introduction

Many interesting parts of mathematics has been discovered and found their uses via many advanced and creative ways of thinking, for example, complex numbers which we will talk about later. However, sometimes the most creative way of thinking is the most simple way, where we use other people’s methods and apply it to your own.

In this paper, we will discuss a topic in maths, that will potentially question the way you think about many areas of maths and potentially lead to another great discovery such as the one we will speak of.

To make sure we are all on the same page, let’s recap some simple yet complicated tactics that was used to bring a potential favourite branch of numbers amongst many people to life.

2 What are Imaginary Numbers?

So, to start off, we all know and love imaginary and complex numbers and the various amount of things we can do with it, ranging from finding extra solutions for polynomials all the way up to the deepest depths of complex analysis. However, to start off, what actually are imaginary numbers. To be put simply, these poorly named numbers can be defined as

$$i^2 = -1$$

Therefore meaning that,

$$i = \sqrt{-1}$$

Now, after defining this interesting number, we ask ourselves ‘how do we actually use this identity to help us solve things in maths?’ I’m sure you already know this if you’ve come to read this, however, if not let this recap the basics if complex numbers.

2.1. Setting the foundations of complex numbers

Let us start by asking ourselves ‘What is a complex number?’ This question can be put to rest with the equation

$$z = a + bi$$

Where z is a complex number, a and b are both constants with ‘a’ being the real part of the complex number, and ‘b’ being the imaginary part, or being the coefficient of the imaginary unit.

An example of this could be shown with $z = 3 + 4i$, where 3 would be the real part therefore meaning 4 must be the imaginary part.

Now, so far, these haven't been too helpful to us, since the definition alone doesn't really help us with much besides potentially solving a few equations we might have thought to previously be unsolvable. This is where something called an **Argand Diagram** came about.

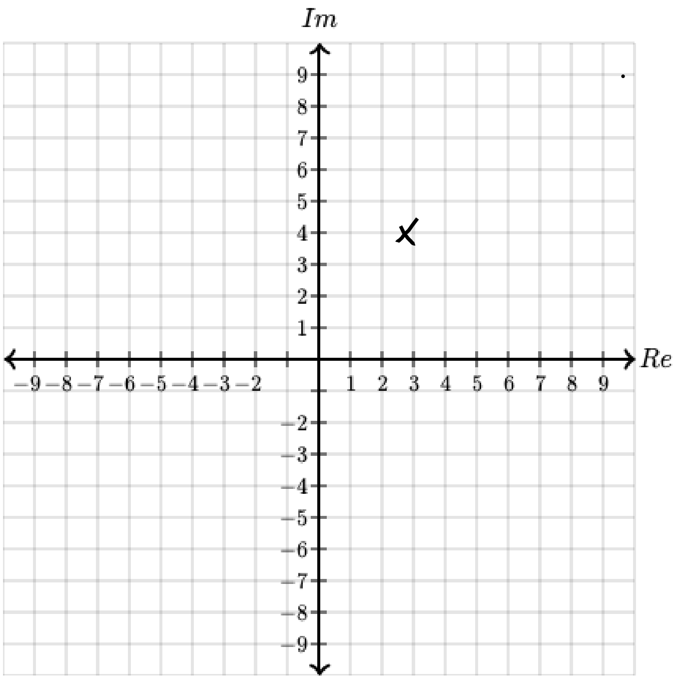

Figure 1: An example of an Argand Diagram with 3+4i plotted

An **Argand Diagram** allows us to represent our complex numbers in a graphical method via the x-axis being the **Real Axis** and the y-axis being the **Imaginary Axis**, meaning that you've essentially got the real number line and created a perpendicular line in the centre, as an imaginary number line.

On its own, yet again this doesn't help much, but with some further definitions the complex world begins to unravel even beyond.

2.2. Modulus and Argument

To extend upon our knowledge let's define **modulus**, which is the distance from the point on the Argand diagram to the centre (0+0i). This property is usually denoted by either 'r' or |z|

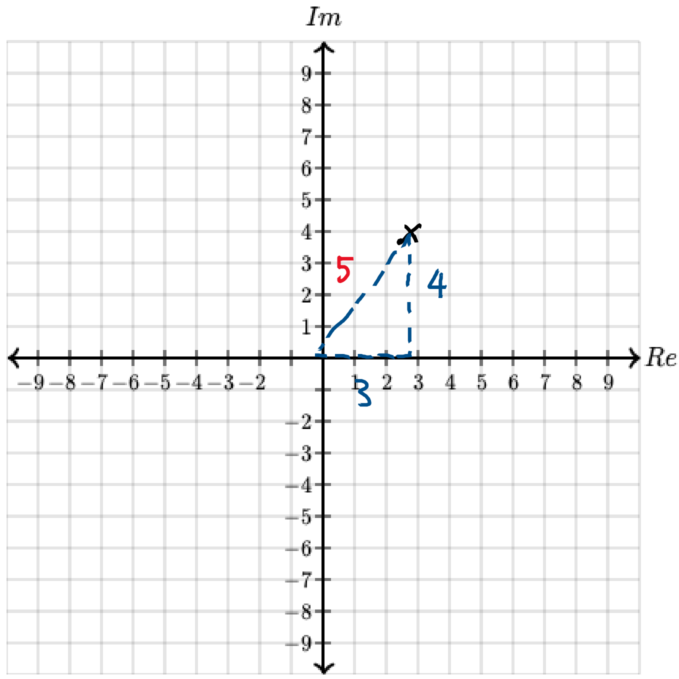

Figure 2: An example of finding the modulus of 3+4i

As can be seen in *Figure 2, the modulus* of a point on an Argand diagram can usually be resolved via the usual Pythagoras we all learnt when we were younger.

In this case, we have found the **modulus** of our point 3+4i by:

$$\sqrt{3^2 + 4^2} = 5$$

Therefore, we can say that in order to find the **modulus** of a complex number can be found using the formula:

$$|z| = \sqrt{a^2 + b^2}$$

Another vital definition to really understanding complex numbers is the **argument** of a complex number, which is easily defined as the angle (usually in radians not degrees) of the point on an Argand diagram relative to the positive real axis (which we remember is the x-axis).

The **argument** of a complex number is usually denoted as arg(z) and is found in a method very similar to the modulus.

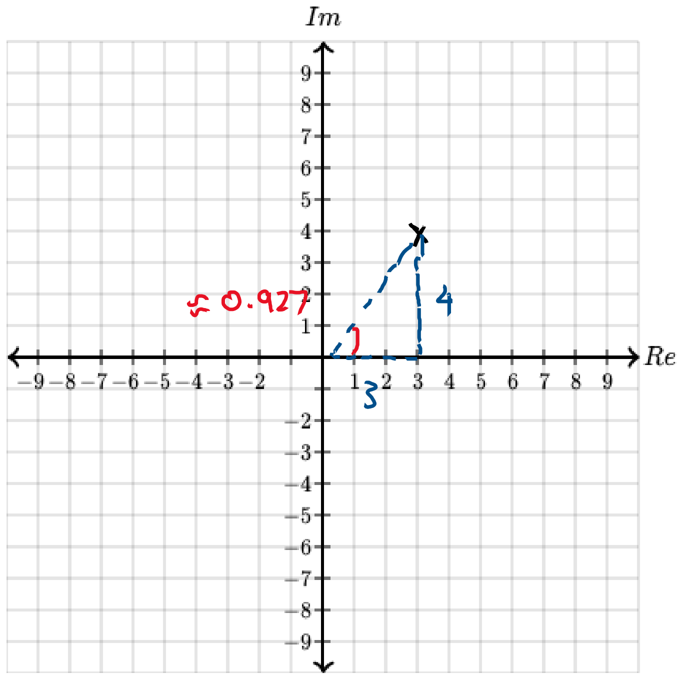

Figure 3: An example of finding the argument of 3+4i

As you can see through *figure 3*, when you produce a right angled triangle again as such, you can find the angle representing the **argument** with simple trigonometry except we convert to radians at the end. The way we would find this particular argument would be:

$$\arctan\left(\frac{4}{3}\right) = 0.927$$

From this method, we can deduce that the formula that provides the argument would be:

$$\arctan\left(\frac{b}{a}\right) = \arg(z)$$

Now that we have defined our two key definitions, we can actually do something very interesting with complex numbers and that is converting them from their usual form into something called **modulus-argument form**. This way of writing complex numbers is something that combines both our **modulus** and our **argument** into something that can completely describe any complex number.

The way that most textbooks explain it, is simply to memorise it, however I think a derivation of the formula could help understand why this works.

In order to do this, we need to use the previous *figure 2 and 3* to visualise a triangle and label the sides accordingly,

The side with the length of 5, we will call the hypotenuse or H

The side with the length of 4, we will call the opposite or O

The side with the length of 3, we will call the adjacent or A

Now if we relate that to our general formula of $a + bi,$

We can convert this into our new names we have given for the sides meaning that the general complex number is now redefined as:

$$z = A + Oi$$

If we divide both sides by H, we get;

$$\frac{z}{H} = \frac{A}{H} + \frac{O}{H}i$$

Using basic trigonometric ratios we can assume that A/H is equal to cosine of the argument (if you use that triangle the angle theta is the same as the argument) and O/H is equal to sine of the argument. This means that:

$$\frac{z}{H} = \cos(\theta) + \text{isin}(\theta)$$

By multiplying both sides by H, which is also the same as the modulus, as seen in *figure 2*, we get the final equation of

$$z = r(\cos(\theta) + \text{isin}(\theta)$$

This form has many uses in later levels of maths such as De Moivre's Theorem, conversion into exponential form and areas of analysis. However, this paper isn't about the wonders of complex numbers so I'm not going to go into too much detail.

2.3. Complex Arithmetic

To leave the simplest bit of complex numbers for last is probably a nice break, don't you think? This is probably the most important part in relation to the main theme of this paper so make sure you know this.

If we have two complex numbers:

$$z = a + bi$$

And,

$$w = c + di$$

When you add these two complex numbers, your output would give you:

$$z + w = (a + c) + (b + d)i$$

Similarly taking complex numbers gives you:

$$z - w = (a - c) + (b - d)i$$

When we multiply two complex numbers we get:

$$zw = ac + cbi + adi + bdi^2$$

Because $i^2 = -1$

$$zw = (ac - bd) + (cb + ad)i$$

In order to continue we need to define something called a **complex conjugate**, which means you have the opposite imaginary part (or on an Argand diagram it's a reflection in the real axis), which is denoted by a bar on top. For example the conjugate of z would be:

$$\bar{z} = a - bi$$

Therefore,

$$z\bar{z} = a^2 + b^2$$

We use this in order to simplify division, which you can see here:

$$\frac{z}{w} = \frac{a + bi}{c + di}$$

$$= \frac{z}{w\overline{w}}$$

$$= \frac{a + bi}{c^2 + d^2}$$

This is helpful because the denominator is now real, and therefore you can write the complex number back into its nice form.

3 The unknown dual number

Now that we have recapped and delved into the incredible universe of imaginary numbers and some of their uses, let's find something the average mathematician would have never heard of.

Say, if I was to ask you 'if a number squared is equal to zero, what would that original number be equal to?', I can imagine that you'd square root both sides and most people would probably say 0 because the square root is equal to 0 always.

Well what if it wasn't? Welcome to dual numbers…

3.1. What is a dual number?

A dual number is a number which squared is equal to zero without itself being equal to zero, and is represented by the Greek letter epsilon, which can be shown by this beautiful equation below:

$$\epsilon^2 = 0$$

Where epsilon itself is not 0.

To be honest, this is the only definition, and so far you're probably thinking "This is rubbish, this contradicts everything" but later on you'll see some very interesting properties this dual number has.

3.2. Dual Arithmetic??

Similar to complex numbers we will now see some properties of dual numbers when different operations act on them.

To start off with we need to define dual numbers again in a form similar to complex numbers where

$$z = a + b\epsilon$$

And

$$w = c + d\varepsilon$$

This assumes that both ‘a’ and ‘c’ are the real parts and ‘b’ and ‘d’ are the dual parts.

When we add these we get,

$$z + w = (a + c) + (b + d)\varepsilon$$

Similarly when we subtract dual numbers we get:

$$z - w = (a - c) + (b - d)\varepsilon$$

When we multiply them something strange happens,

$$zw = ac + (bc + ad)\varepsilon + bd\varepsilon^2$$

Because of the definition of a dual number, the ε squared becomes zero:

$$= ac + (bc + ad)\varepsilon$$

Before we carry on we need to have our dual number conjugate which similar to imaginary numbers, the dual number conjugate for z will be:

$$\bar{z} = a - b\varepsilon$$

Therefore,

$$z\bar{z} = a^2$$

When we divide anything by ε by itself it is undefined because you’ll end up having to divide by 0 because of the following:

$$\frac{x}{\varepsilon} = \frac{(x\varepsilon)}{\varepsilon^2}$$

$$= \frac{x\varepsilon}{0}$$

And as we know dividing by 0 is like breaking the laws in mathematics therefore meaning that you cannot decide by epsilon by itself.

However, unbelievably without seeing proof, you can divide by dual numbers when a, which is the real part is not equal to 0 because when you multiply by its conjugation, you'll end up dividing it by a squared.

This is a very interesting property of dual numbers which separates it from dual numbers and makes it more similar to zero without being zero itself, yet there's one property in particular that strikes my eye when researching into dual numbers.

3.3. Automatic Differentiation

Although the dual number seems very interesting, so far we haven't seen a very helpful use of it, until now where I'll show you how some computers use dual numbers to differentiate very quickly, compared to other methods we may use.

Say we have a function:

$$f(x) = 3x^2$$

And I asked you to find me f(3) and f'(3) you would probably substitute 3 for x, which would let you obtain 27, them you would differentiate the function to get 6x, then substitute 3 in again to get 18 giving your 2 answers like that. Now, I don't disagree with you but I'm going to show you another way of finding these two values in one go:

$$f(3+\varepsilon) = 3(3+\varepsilon)^2$$

$$= 3(9+6\varepsilon)$$

$$= 27+18\varepsilon$$

Now what you might notice when we found f(3+ε) is that it gave us f(3) in the real part of the answer and f'(3) in the dual number part of the answer. This mysterious property has very few proofs and even those proofs are very complicated and not worth delving into in this short paper however they all show us that,

$$f(x+\varepsilon) = f(x) + f'(x)\varepsilon$$

This relationship between differentiation and dual numbers is a very scary thing in mathematics and could hold unlimited potential but only time will tell what mysteries can be found with the darkest depths of dual numbers.

4 Conclusion

So, we have now briefly looked at the wonders of dual numbers and unfortunately due to the lack of knowledge and recognition of this topic there's not much more, however I am recently working on many theories that could be looked into such as matrices, number theory and its own unique analysis.

I hope you have enjoyed learning about this and hopefully you have learned something new and interesting, thank you for reading.