

CUT PASTE DUPLICATE: THE BIZARRE BANACH-TARSKI PARADOX

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1 Introduction

Imagine holding a solid ball. Now imagine yourself cutting it into five pieces, not by slicing it apart with a kitchen knife, but with the surgical precision tool - Mathematics. Then by just rotating and moving the pieces around (no squeezing, no stretching and definitely no wizardry), you rearrange those pieces into two different balls - now here is the catch, the two balls are absolutely IDENTICAL TO THE ORIGINAL! This sounds like a magician's trick, and it most definitely is, with math doing the magic. Welcome to the Banach-Tarski Paradox, and let us discuss how the paradox works by diving deep into the mathematical ideas behind it, such as the axiom of choice, group theory and more. After that, we will understand the implications of this paradox in how we perceive space, logic and the very nature of mathematics.

2 Statement of Paradox

This Paradox was formulated back in 1924, over a 100 years ago, by Stefan Banach and Alfred Tarski, two Polish mathematicians. The paradox is defined as follows:

Let $B \subset \mathbb{R}^3$ be a solid ball. Then there exist disjoint subsets $A_1, A_2, A_3, A_4, A_5 \subset B$ such that each A_i is non-measurable,

and there exist rotations and translations T_1, T_2, T_3, T_4, T_5 such that:

$$T_1(A_1) \cup T_2(A_2) \cup T_3(A_3) = B \text{ and } T_4(A_4) \cup T_5(A_5) = B$$

However, let us try to understand this in simpler words first. What the paradox essentially states is that:

*“A solid ball in three-dimensional Euclidean space can be decomposed into a finite number of disjoint subsets, which can then be reassembled, using only rotations and translations, into **two** solid balls identical to the original.”*

We are breaking a ball into five different mathematical pieces, and then individually rearranging the pieces to reassemble two of the initial balls without adding any material. Fascinating right? Now let us understand the mathematical ideas behind this.

3 The Mathematical Machinery

3.1 The Axiom of Choice

The Axiom of choice is what makes this paradox possible. It is formally defined as:

$$\forall F [(\forall A \in F, A \neq \emptyset) \Rightarrow \exists f: F \rightarrow \cup F \text{ such that } \forall A \in F, f(A) \in A]$$

This is a lot to digest, especially if one is unknown to advanced mathematics. So let us break the axiom of choice down more intuitively. Let us say that you have an infinite number of baskets. Each basket has at least 1 item inside, it can be anything, an apple, a banana, a pen etc. What the axiom of choice states is that even without a rule for how to pick items from the baskets, it is possible to pick 1 item from each basket. This seems intuitively impossible at first, however with the axiom of choice as a tool, we can just assume that we are able to carry out such an act. This is a key building block of the Banach-Tarski Paradox and the whole foundation of the paradox is built upon the axiom of choice.

3.2 Non-Measurable Sets

Non-measurable sets are sets that cannot be assigned with a constant volume, area or length. It is impossible to assign a physical measure to the set, and under the axiom of choice, they can be constructed and used in paradoxes like the Banach-Tarski, as the “parts” that the sphere in the

Banach-Tarski is cut into, are in-fact non-measurable sets. The rationale behind using non-measurable sets is that they allow us to form conjectures that are mathematically “legal”, even if they may not be physically possible.

3.3 Free Groups

This is where we start to consider abstract algebra. First let us understand what a group is. A group in Mathematics is a set with an operation (like multiplication or addition) that satisfies four properties:

1. **Closure:** Doing the operation on 2 elements gives rise to another element in the group
2. **Associativity:** The order of the grouping does not matter,
3. **Identity:** There exists an element e where $e \cdot a = a \cdot e = a$
4. **Inverses:** Every element has an inverse a^{-1} such that $a \cdot a^{-1}$ gives e .

So what makes a free group? A free group is a group with the fewest possible rules. We start out with generators (a and b), and we can combine them in any way we want, including their inverses, we can form as many sequences as possible, and the only rule that exists is that an element multiplied by its inverse becomes the identity and is thus cancelled. In the Banach-Tarski paradox, we consider a free group with 2 generators, which is written as:

$$F_2 = \langle a, b \mid \text{no relations} \rangle$$

Intuitively, we can think of it as the group of all possible “words” formed from the “letters” a , b , a^{-1} and b^{-1} , where the only rule is that adjacent inverse pairs get cancelled. Thus, we can see that there are infinitely many words that are able to be formed.

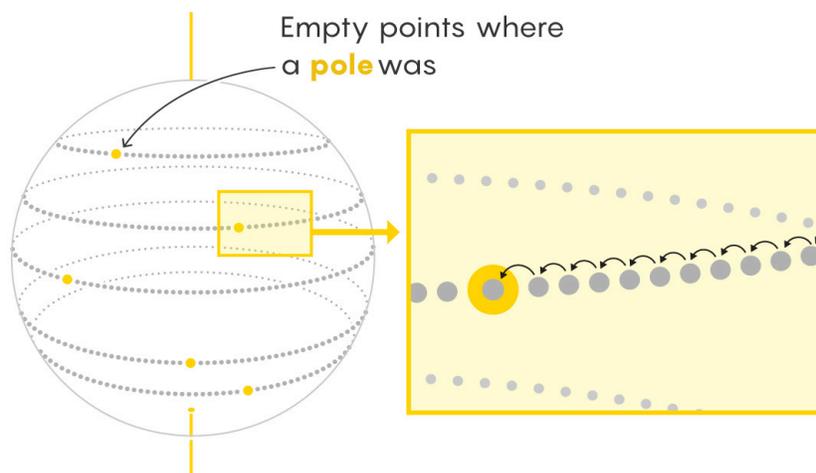
Before diving into what exactly the free group does, let us understand what a 2-Sphere is. A 2-sphere is a 2 dimensional sphere embedded into a 3-dimensional space, it is kind of like the surface of a perfectly round ball. The formal definition of the 2-sphere is given by:

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

This is important as the Banach-Tarski begins with the surface of a solid 3D ball, which is essentially a 2-Sphere and then eventually the structure is extended to fill the entire ball.

The F_2 group acts on the 2-Sphere S^2 by performing a specific set of rotations about the two axes, which creates orbits of points on the sphere which are related by these rotations. Then, by

using the axiom of choice, we choose one point from each orbit, which forms a set. This set forms the core of the non-measurable set that we will use to construct the paradox.



(The image above shows how the orbits appear on the 2-Sphere)

4 The Construction

Before we dive into the steps on exactly how we can construct two spheres from one, let us understand the concept of amenability. A group is called amenable if it has a way of assigning a measure to its subsets in a way the group's structure is preserved. We can think of this measure as kind of like volume, or area, or length - as it is additive and consistent under transformations.

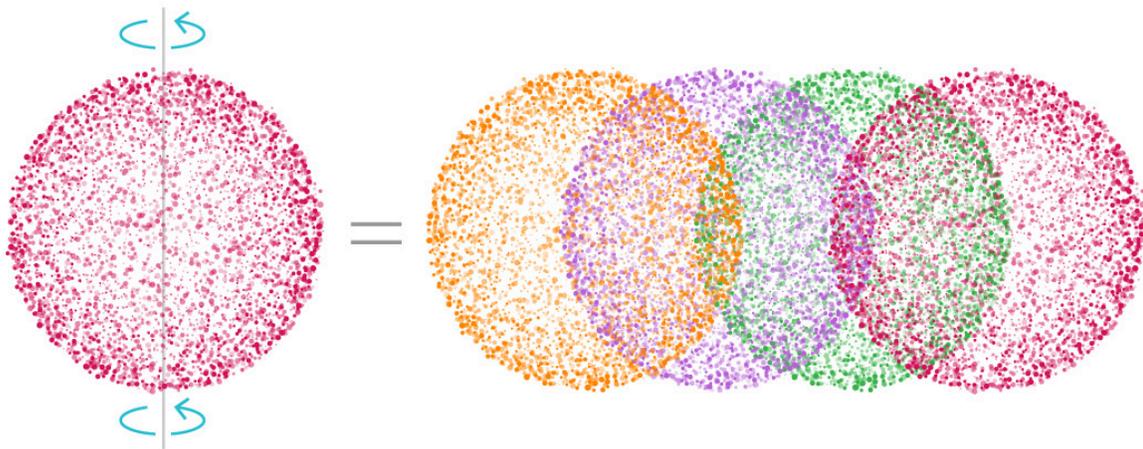
On the contrary, non-amenable groups (like free groups such as F_2) do not have such a measure. This lack of a volume-like measure means that it is possible to rearrange the elements of the group in extremely bizarre ways without violating any laws, and this idea is what enables paradoxical decompositions like splitting a set into parts, and assembling them into something seemingly bigger. Let us try to visualise this as a real world analogy, which is quite commonly used in explaining non-amenable sets.

Imagine that people are trying to divide a cake, but the people follow a set of very chaotic and unreasonable rules, for example: "I only want a slice, if someone has the exact opposite one" or like "if Alice wants a particular piece then I want three pieces, one of which must include the piece Alice's piece wants" and infinitely many other absurd rules. Thus we try to divide the cake to meet everyone's rules, it is impossible as someone ends up duplicating, cancelling or demanding another slice that makes the configuration impossible. Thus it becomes impossible to

assign consistent sizes to each piece that respects the group's structure. I hope this analogy allows a better visualisation of how non-amenable groups work.

Now let us walk through the high level steps of the paradoxical decomposition:

1. *Start of with a solid ball in \mathbb{R}^3 .*
2. *Define a group action of F_2 on the surface of the ball.*
3. *Partition the sphere into orbits under this group action.*
4. *Use the axiom of choice to select a representative point from each orbit.*
5. *Form five different disjoint subsets based on these orbits and group elements. (usually known as the East group, West group, South group, North group, and the group of starting points)*
6. *Apply appropriate rotations and translations to assemble two full copies of the original ball.*



Rotating the **East group** to the **West** duplicates the **South**, the **North**, the **starting points**, and the **East group** itself.

The image above shows how different sets of points can be extracted from a singular solid sphere after rotations, which leads to overlaps between the groups. The elements of this overlap are

essentially what form the other sphere, and via precise group actions, and also the axiom of choice, the overlap is exactly equal to the size of the initial sphere. This causes the overlapped points to form one sphere, and the rest of the remaining points to form the other sphere, both identical to the initial starting sphere.

5 Why does this not work in Real life

Now, after figuring out how the construction works, you may already know why this is not able to happen in real life, but only in the mathematical world. In real life, physical objects are made of atoms and each object only has a finite number of atoms, thus it is impossible to divide the objects into infinitely many points.

The key idea is that the Banach-Tarski pieces are not ordinary chunks of matter. They are abstract sets of points that do not have a definite volume at all, i.e they are non-measurable sets, and their construction is only possible using a controversial yet powerful mathematical tool, the axiom of choice.

The Banach-Tarski paradox may seem extremely shocking and disturbing at first as us humans have the idea that cutting something into pieces and reassembling it should not increase its volume, however, if we are willing to give up assumptions and think outside the box (which is physical reality in this case), and accept some other conditions (like the axiom of choice), then it is fairly easy to construct such puzzling sequences, which ultimately challenges the boundaries of mathematics as a whole.

This paradox deeply highlights the difference in constructive and non-constructive mathematics, where we can prove something exists but we are never able to describe it. I guess one could call it the mathematical equivalent of having a friend saying that he built a teleporting machine, but it's invisible, untouchable and only works in theory.

6 Philosophical Implications

The Banach-Tarski paradox forces us to confront some tough philosophical questions:

1. Is modern mathematics more about describing reality, or building logical systems that may work but not in our physical reality?
2. Should we accept axioms and use them to further develop mathematics even though they lead to very bizarre consequences?

3. Can we ever separate the Truth and Constructability, and distinguish them clearly?

While constructivist mathematicians would say NO to embracing the axiom of choice, and only accepting proofs that can actually construct the objects being claimed, many others find the paradox a thrilling, yet understandable demonstration of the power that abstraction provides us because let's face it: The idea that we can clone a sphere without magic or machines is just too cool to disregard!

7 Conclusion

The Banach-Tarski paradox is not a recipe for infinite oranges, nor can you double your money overnight. However, it is a puzzling result that is obtained from perfectly sound mathematics. It highlights the power of modern mathematics, and reminds us that mathematics is not only the language of the universe but also the mindmap of our imagination. This essay was an introduction to the mathematical tools behind the paradox, the steps behind how the paradox is constructed, as well as a look into the implications of such a paradox existing in our mathematical world.

8 References

Images:

<https://www.quantamagazine.org/how-a-mathematical-paradox-allows-infinite-cloning-20210826/>

How I first came across the paradox:

<https://www.youtube.com/watch?v=s86-Z-CbaHA>

https://en.wikipedia.org/wiki/Banach%E2%80%93Tarski_paradox

How I dove deeper into the formal math behind the paradox:

<https://math.uchicago.edu/~may/REU2014/REUPapers/Robinson.pdf>

9 Author's Note

Having only recently learnt about this topic, I am very glad to have written a complete essay on a fascinating topic, that I think can be understood by the average person, and does not require a large amount of background in mathematics. While I have always been a keen reader of mathematics articles, fascinated by mathematical research, and a problem solver, I had never imagined writing my own essay/mini-article and I would like to thank Dr Tom Crawford and the organizers of the Tom Rocks Maths Essay competition for providing me this platform. I would also like to thank my teachers for instilling the love of mathematics in me at a young age. I hope you had an enjoyable time reading this essay, and I hope to continue writing more of such essays on other interesting topics in mathematics that can be accessible to and understood by the public.

~Atharv

